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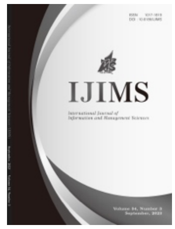

An Optimal Policy for Weibull Distribution of Deteriorating Items with Backlogging and Ramp-type Demand Under Inflation

Hui-Ling Yang[1], *Chun-Tao Chang*[2], *Yao-Ting Tseng*[3*]
[1] Department of Intelligent Technology and Application, Hung Kuang University, Taiwan
[2, 3] Department of Statistics, Tamkang University, Taiwan

Keywords

Ramp-type; Deterioration; Backlogging; Limited Storage Capacity; Inflation

Abstract

Generally, a high-tech product's demand rate during the growth stages increases significantly with linear or exponential growth, and then, in the maturity stage, it remains almost the same. This is a so-called ramp-type demand rate. Additionally, a specific product may deteriorate over time. The more deterioration there is, the higher the order quantity. Based on this consideration, the deterioration rate could not be ignored. Therefore, this paper established a two-warehouse partial backlogging inventory model incorporating a ramp-type demand for three-parameter Weibull distribution deteriorating items. The main task is to derive an optimal replenishment strategy that minimizes the net present value of the total relevant cost per unit of time. The results of the proposed inventory system are verified through numerical examples and sensitivity analysis. The numerical result offers a reference for inventory managers to reasonably order quantities when facing a ramp-type demand rate with Weibull distributed deterioration.

1. Introduction

Researchers have extensively investigated the economic order quantity (EOQ) model, focusing on scenarios where the demand rate remains constant. However, in real life, just like fashion products or new mobile phones, the demand for them varies over time, such as linear, ramp-type, exponential, price, stock, or expiration date dependent. On the other hand, some products, such as metal or stone, have zero or near zero deterioration rates, but not others, such as vegetables, fruits, or electronic appliances. Their deterioration varies with time, such as constant, exponential, or expiration date. From the perspective of these two factors, this manuscript will concentrate on ramp-type demand rates and the deterioration modeled by the three-parameter Weibull distribution. Related papers are demonstrated as follows.

*corresponding author. Email: yaoting.tseng@mail.tku.edu.tw

1.1 Paper related to ramp-type demand

In real life, for many new products, such as high-tech products or new brands of customer goods launched on the market, or seasonal products, their demand rates increase significantly during the linear or exponential growth phase and remain near constant in the maturity stage. The term "ramp type" represents such a demand pattern. Manna and Chaudhuri (2006) formulated an EOQ model accommodating a ramp-type demand rate, encompassing time-dependent deterioration and shortages. Deng et al. (2007) highlighted inventory models addressing deteriorating items featuring ramp-type demand rates. Next onwards, Panda et al. (2008) studied an optimal replenishment policy for perishable seasonal products in a season with ramp-type dependent demand. Agrawal and Banerjee (2011) established a two-warehouse inventory model incorporating ramp-type demand and partially backlogged shortages. Skouri et al. (2011) outlined inventory models incorporating ramp-type demand rates, time-dependent deterioration rates, unit production costs, and shortages. Agrawal et al. (2013) developed an inventory model for a two-warehouse system that accounts for deteriorating items, ramp-type demand, and partially backlogged shortages. Then, Halim (2017) posed a Weibull distributed distributing inventory model with ramp-type demand rate and full backlogging shortages. Shi et al. (2019) introduced optimal ordering policies for a single deteriorating item with a ramp-type demand rate, accounting for permissible delay in payments. In the meantime, Yang (2019) constructed an inventory model addressing ramp-type demand, integrating two-level trade credit financing tied to order quantity. Hasan et al. (2021) introduced an EOQ model applicable to non-instantaneous deteriorating items with ramp-type demand within a two-warehouse setting. Kumar et al. (2022) formulated a two-level storage inventory model addressing ramp-type demand in inflationary conditions, incorporating partial backordering. Sethi et al. (2022) developed a fuzzy inventory model for deteriorating items with ramp-type demand and shortages in a two-warehouse setup.

1.2 Paper related to two-parameter Weibull distribution

Furthermore, deterioration refers to the degeneration or damage of items in storage. Sometimes, deterioration may lead to complete decay, rendering the product completely unusable and then turned into scrap. To this end, better preservation facilities can reduce the deterioration rate of the product and start the deterioration rate later than normal. Therefore, deterioration is a significant factor in analyzing inventory systems. That's why thinking about deterioration is inevitable. Specific products such as seasonal foods, vegetables, and fruits deteriorate over time during their regular storage period. The degree of deterioration depends on time (linearly, quadratic, or exponential). However, certain items like steel, glassware, hardware, and toys exhibit a minimal rate of deterioration, requiring little consideration for the effect of deterioration when determining the inventory lot size. On the other hand, some items degrade over time, such as battery leakage failure, drug expiration, and so on. Failure rates increase with prolonged periods of item non-utilization.

Early researchers, such as Covert and Philip (1973) devised an inventory model specifically designed for items undergoing deterioration, wherein the degradation rates fluctuate, and they utilized the two-parameter Weibull distribution to model the degradation behavior. Misra (1975) employed the two-parameter Weibull distribution to match the deterioration rate in the production lot-size model. Later, the EOQ inventory model developed by Wu (2001) accommodated items undergoing deterioration characterized by a Weibull distribution alongside a ramp-type demand rate and partial backlogging. Deng (2005) produced an improved inventory model with ramp-type demand and Weibull deterioration. Again, Wee et al. (2005) constructed a two-warehouse inventory model to address deteriorating items characterized by a two-parameter Weibull distribution and a constant partial backlogging rate in the presence of inflation. Lo et al. (2007) developed an integrated production-inventory model incorporating imperfect production processes in an inflationary environment, where product deterioration followed a two-parameter Weibull distribution. Panda et al. (2007) investigated an EOQ model featuring generalized ramp-type demand and deterioration following a Weibull distribution. Skouri and Konstantaras (2009) explored inventory models at the order-level for deteriorating seasonal or fashionable products, integrating a two-parameter Weibull deterioration rate, time-dependent demand, and shortages. Skouri et al. (2009) simultaneously investigated an inventory model incorporating a general ramp-type demand rate, a two-parameter Weibull deterioration rate, and partial backlogging, applying two replenishment policies. Then, Mandal (2010) proposed an EOQ inventory model designed specifically for deteriorating items characterized by Weibull distribution, ramp-type demand, and shortages. Chołodowicz and Orłowski (2021) devised a fresh hybrid discrete-time perishable inventory model relying on Weibull distribution, integrating time-varying demand through a system dynamics approach. Bankole et al. (2022) explored the EOQ approach for perishable goods characterized by a Weibull distribution and an exponential demand rate correlated with price.

1.3 Paper related to three-parameter Weibull distribution

The practical utility of the two-parameter Weibull distribution in real-world contexts may be limited in certain cases. The deterioration of some items commences only after they have been stored for a specific period rather than immediately upon storage. Therefore, a three-parameter Weibull distribution proves to be more applicable in practical scenarios. Philip (1974) expanded upon the model that Covert and Philip (1973) proposed by incorporating a three-parameter Weibull distribution. Chakrabarty et al. (1998) further developed Philip's model (1974) to include shortages and linear trend demand. Giri et al. (2003) improved an EOQ model by incorporating a three-parameter Weibull deterioration distribution, shortages, and ramp-type demand. Later, Yang (2012) established a two-warehouse inventory model with a three-parameter Weibull distribution deterioration under inflation in which shortages are partial backlogging. Bhunia and Shaikh (2014) proposed a deterministic inventory model for deteriorating items with demand influenced by selling price and a three-parameter Weibull distributed deterioration. Chakraborty et al. (2018) formulated a two-warehouse partial backlogging model featuring a ramp-type demand rate and deterioration described by a three-parameter Weibull distribution, considering inflation and permissible delay in payments.

Then, Shaikh et al. (2019) devised an inventory model for a deteriorating item with a three-parameter Weibull distribution, incorporating variable demand influenced by price and frequency of advertisement under trade credit. The above property can be characterized in Table 1.

Table 1. *Major Characteristic of Inventory Models on Selected Articles*

Reference	Demand Pattern	Limited storage capacity	Deterioration	Backlogging	Discounted Cash-flow
Agrawal and Banerjee (2011)	Ramp-type	2W	No	Partial	No
Skouri et al. (2011)	Ramp-type	No	Time-dependent	No	No
Yang (2012)	Constant	2W	Three-parameter Weibull distributed	Partial	Yes
Agrawal et al. (2013)	Ramp-type	2W	Yes	Partial	No
Bhunia and Shaikh (2014)	Price-dependent	No	Three-parameter Weibull distributed	No	No
Halim (2017)	Ramp-type	No	Two-parameter Weibull distributed	Full	No
Chakraborty et al. (2018)	Ramp-type	2W	Three-parameter Weibull distributed	Partial	No
Wu et al. (2018)	Constant	No	Yes		No
Shaikh et al. (2019)	Price and frequency of advertisement-dependent	No	Three-parameter Weibull distributed	No	No
Shi et al. (2019)	Ramp-type	No	Yes	No	No
Yang (2019)	Ramp-type	No	No	No	No
Chołodowicz and Orłowski (2021)	Time-varying dependent	No	Two-parameter Weibull distributed	No	No
Hasan *et al.* (2021)	Ramp-type	2W	Yes	No	No
Bankole *et al.* (2022)	Price-dependent	No	Two-parameter Weibull distributed	No	No
Kumar *et al.* (2022)	Ramp-type	2W	No	Partial	Yes
Sethi *et al.* (2022)	Ramp-type	2W	Two-parameter Weibull distributed	Partial	No
This paper	Ramp-type	2W	Three-parameter Weibull distributed	Partial	Yes

Remember that (i) the three-parameter Weibull distribution reduces to a two-parameter Weibull distribution when the location parameter is zero. (ii) The Weibull distribution transforms exponentially When the shape parameter equals 1. Hence, the three-parameter Weibull distribution provides a more generalized framework for analysis.

Therefore, contrary to the above articles, the proposed model deals with the demand rate as a ramp-type demand rate rather than a constant. This demand rate is different from that introduced by Yang (2012). The proposed model is a two-warehouse partial backlogging inventory model under inflation, which incorporates both the ramp-type demand rate and the three-parameter Weibull distribution deterioration. The deterioration distributions of the two warehouses are assumed to be independent. Several situations are discussed based on the numerical relationship between parameters and decision variables. The study's findings reveal a distinct and singular optimal solution for each case. Finally, it supplies numerical illustrations for clarity and conducts sensitivity analysis on various parameters.

2. Assumptions and notation

2.1 Assumptions

Yang (2012) follows the assumptions used in this paper. The reader can refer to them for details.

2.1 Notation

Similarly, Yang (2012) follows the notation used in this paper except for the following.

$D(t)$ = the demand rate at time t. we assume that $D(t)$ is constant and deterministic after the length of demand growth stage μ (in years), and $D(t)$ is an increasing linear function of time t during the growth stage. That is,

$$D(t) = \begin{cases} f(t) & t < \mu \\ f(\mu), & t \geq \mu \end{cases}, \text{ where } f(t) = a + bt,\ a > 0,\ b > 0.$$

c_d = the deteriorated item cost per unit.

t_r = the time at which the inventory level reaches zero in RW, where $t_r > 0$, and is defined as the stock period in RW. We here assume $\gamma_r < t_r$.

t_o = the time at which the inventory level reaches zero in OW, where $t_o > t_r$, and is defined as the stock period in OW. Without loss of generality, we here assume $t_o > \mu$.

T = the replenishment cycle, when the shortage level reaches the lowest point in the replenishment cycle, where $T > t_o$.

TC_{1i} = the present value of the total relevant cost per unit time for cases $\mu \leq t_r$ where i =1, 2, 3, 4.

TC_{2i} = the present value of the total relevant cost per unit time for cases $\mu \geq t_r$ where i =1, 2, 3.

C_{HR} = the present value of the inventory holding cost in RW.

C_{DR} = the present value of the cost for the deteriorated items in RW.

C_{HO} = the present value of the inventory holding cost in OW.

C_{DO} = the present value of the cost for the deteriorated items in OW.

C_B = the present value of the backlogging cost in (t_o, T).

C_L = the opportunity cost due to lost sales in (t_o, T).

3. Research models

Two cases are to be considered for each model introduced by Yang (2012). However, in this paper, we only discuss the case of Model 1, in which the shortages are allowed at the end of the replenishment cycle, not at the beginning. Because of many fashionable or high-tech electronic products, their demand rates are increasing significantly, and shortages cannot be permitted.

For the research model, at time $t = 0$, many specific units enter the system, and a portion is used to meet the partially backlogged items towards previous shortages. At initial, there are S units remain in the system, W units are kept in OW, and the rest ($S - W$ units) are stored in RW. OW goods are consumed only after the inventory goods are consumed in the rented warehouse.

Due to the numerical relation between the parameters and decision variables, the model can be proposed in various cases. The details are shown in Figure 1.

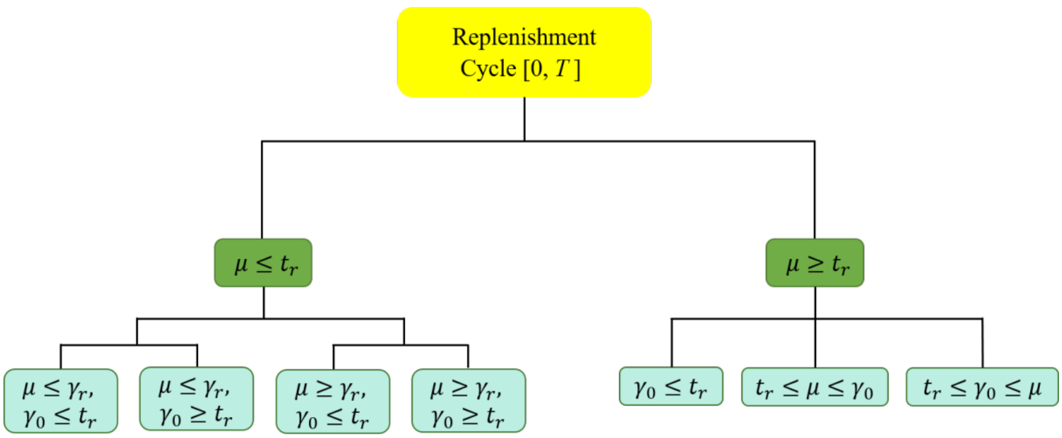

Fig 1. *Graphical Illustration of Various Cases of The Proposed Model*

Next, we first consider the numerical relation between the parameter μ and the decision variable t_r, since the goods are depleted first in the rented warehouse. There are two parts to be discussed (i) $\mu \le t_r$ and (ii) $\mu \ge t_r$

3.1 Part 1: $\mu \leq t_r$

In RW, during $(0, t_r)$, two cases need to be discussed: (i) $\mu \leq \gamma_r$ (ii) $\mu \geq \gamma_r$. Furthermore, in OW, during $(0, t_o)$, depending on the relation between γ_o and t_r, there are also two cases to be considered, (i) $\gamma_o \leq t_r$ (ii) $t_r \leq \gamma_o$. Therefore, in the replenishment of this part, there are four cases to be discussed: (1) $\mu \leq \gamma_r$, $\gamma_o \leq t_r$ (2) $\mu \leq \gamma_r$, $t_r \leq \gamma_o$ (3) $\mu \geq \gamma_r$, $\gamma_o \leq t_r$ (4) $\mu \geq \gamma_r$, $t_r \leq \gamma_o$.

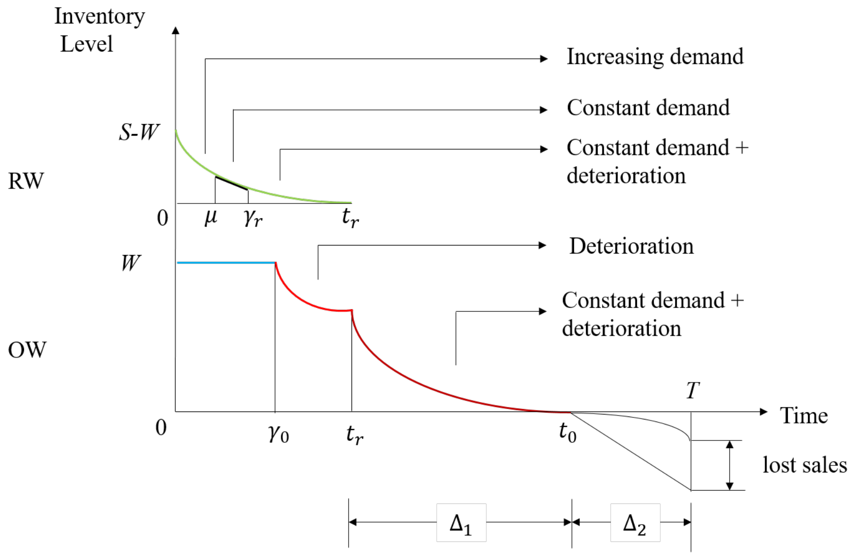

Fig 2. *Graphical Representation of The Research Model (where* $\mu \leq \gamma_r$ *and* $\gamma_0 \leq t_r$*)*

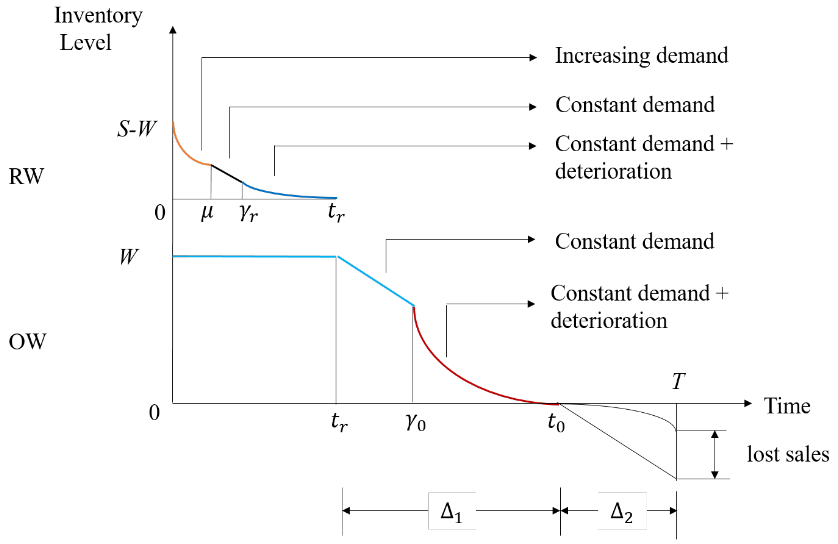

Fig 3. *Graphical Representation of The Research Model (where* $\mu \leq \gamma_r$ *and* $\gamma_0 \geq t_r$*)*

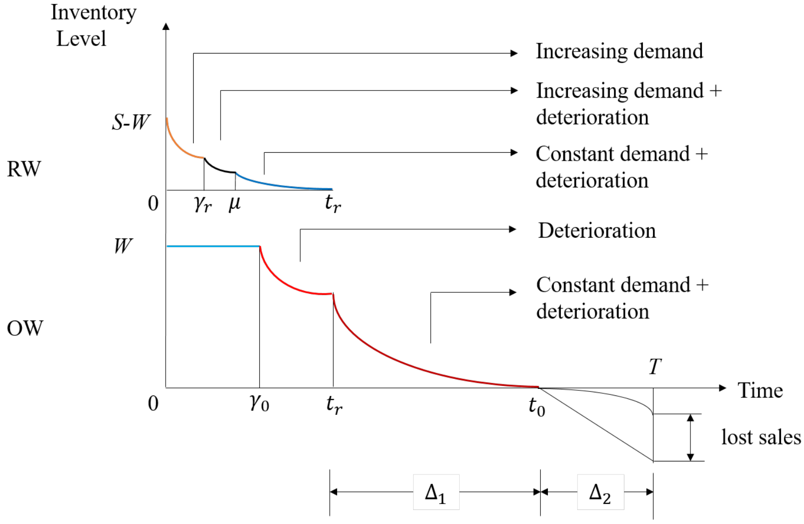

Fig 4. *Graphical Representation of The Research Model (where $\mu \geq \gamma_r$ and $\gamma_0 \leq t_r$)*

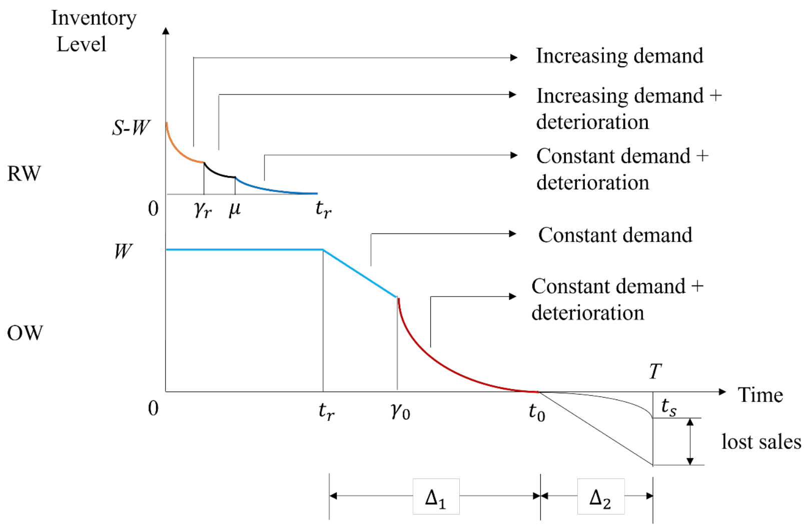

Fig 5. *Graphical Representation of The Research Model (where $\mu \geq \gamma_r$ and $\gamma_0 \geq t_r$)*

Now, we first discuss the model formulated in a rented warehouse.

3.1.1 The model was formulated in a rented warehouse

Case 1. $\mu \leq \gamma_r$

There is no deterioration, and the inventory gradually decreases due to demand in $(0,\mu)$, where $0 \leq t \leq \mu$, after the changing point μ, the inventory level depletes due to constant demand in (μ,γ_r), where $\mu \leq t \leq \gamma_r$. The inventory level at time t is governed by the following differential equation:

$$\frac{dI_r(t)}{dt} = -f(t), \qquad 0 \le t \le \mu, \tag{1.1}$$

$$\frac{dI_r(t)}{dt} = -f(\mu), \qquad \mu \le t \le \gamma_r, \tag{1.2}$$

with the boundary condition $I_r(0) = S - W$. During (γ_r, t_r), the inventory depletes due to both constant demand and deterioration, and it vanishes at $t = t_r$; it satisfies the following differential equation:

$$\frac{dI_r(t)}{dt} = -f(\mu) - Z_r(t)\, I_r(t), \qquad \gamma_r \le t \le t_r, \tag{1.3}$$

with the boundary condition $I_r(t_r) = 0$. The solutions to (1.1)-(1.3) are as follows.

$$I_r(t) = S - W - \int_0^t f(v)dv, \qquad 0 \le t \le \mu, \tag{2.1}$$

$$I_r(t) = S - W - \int_0^\mu f(v)dv - f(\mu)(t-\mu), \qquad \mu \le t \le \gamma_r, \tag{2.2}$$

and

$$I_r(t) = f(\mu)e^{-\alpha_r(t-\gamma_r)^{\beta_r}} \int_t^{t_r} e^{\alpha_r(v-\gamma_r)^{\beta_r}} dv, \qquad \gamma_r \le t \le t_r, \tag{2.3}$$

respectively. Using the continuity of $I_r(t)$ at time $t = \mu$ and $t = \gamma_r$, from (2.1)-(2.3), we have

$$S = W + \int_0^\mu f(t)dt + f(\mu)(\gamma_r - \mu) + f(\mu)\int_{\gamma_r}^{t_r} e^{\alpha_r(t-\gamma_r)^{\beta_r}} dt. \tag{3}$$

The amounts of deteriorated items in RW is

$$f(\mu)\int_{\gamma_r}^{t_r}\left(e^{\alpha_r(t-\gamma_r)^{\beta_r}} - 1\right)dt. \tag{4}$$

Thus, the cumulative inventory in RW during $(0, t_r)$ is $\int_0^{t_r} I_r(t)dt$. Therefore, $C_{HR_{11}}(0, t_r) = c_{hr}\int_0^{t_r} e^{-rt} I_r(t)dt$

$$= c_{hr}\left\{\begin{array}{c} \int_0^\mu e^{-rt}\left(S - W - \int_0^t f(v)dv\right)dt \\ + \int_\mu^{\gamma_r} e^{-rt}\left[S - W - \int_0^\mu f(t)dt - f(\mu)\ (t-\mu)\right]dt \\ + f(\mu)\int_{\gamma_r}^{t_r} e^{-rt} e^{-\alpha_r(t-\gamma_r)^{\beta_r}} \int_t^{t_r} e^{\alpha_r(v-\gamma_r)^{\beta_r}} dv\, dt \end{array}\right\}. \tag{5}$$

And

$$C_{DR_{11}}(\gamma_r, t_r) = c_d f(\mu)\int_{\gamma_r}^{t_r} e^{-rt}\left(e^{\alpha_r(t-\gamma_r)^{\beta_r}} - 1\right)dt. \tag{6}$$

Case 2. $\mu \ge \gamma_r$

There is no deterioration, and the inventory gradually decreases due to demand in the $(0, \gamma_r)$, where $0 \le t \le \gamma_r$, after the deterioration point γ_r, the item begins to deteriorate, and the inventory also gradually decreases due to demand in (γ_r, μ), where $\gamma_r \le t \le \mu$. The inventory level at time t is governed by the following differential equation:

$$\frac{dI_r(t)}{dt} = -f(t), \qquad 0 \le t \le \gamma_r, \tag{7.1}$$

$$\frac{dI_r(t)}{dt} = -f(t) - Z_r(t)\, I_r(t), \qquad \gamma_r \le t \le \mu, \tag{7.2}$$

with the boundary condition $I_r(0) = S - W$. During (μ, t_r), the inventory depletes due to both constant demand and deterioration and it vanishes at $t = t_r$, it satisfies the following differential equation:

$$\frac{dI_r(t)}{dt} = -f(\mu) - Z_r(t)\, I_r(t), \qquad \mu \le t \le t_r, \tag{7.3}$$

with the boundary condition $I_r(t_r) = 0$. The solutions to (7.1)-(7.3) are as follows.

$$I_r(t) = S - W - \int_0^t f(v)dv, \qquad 0 \le t \le \gamma_r, \tag{8.1}$$

$$I_r(t) = S - W - \int_0^{\gamma_r} f(v)dv - \int_{\gamma_r}^{\mu} e^{\alpha_r(t-\gamma_r)^{\beta_r}} f(t)dt + e^{-\alpha_r(t-\gamma_r)^{\beta_r}} \int_t^{\mu} e^{\alpha_r(v-\gamma_r)^{\beta_r}} f(v)dv, \qquad \gamma_r \le t \le \mu, \tag{8.2}$$

and

$$I_r(t) = f(\mu) e^{-\alpha_r(t-\gamma_r)^{\beta_r}} \int_t^{t_r} e^{\alpha_r(v-\gamma_r)^{\beta_r}} dv, \qquad \text{b} \qquad \mu \le t \le t_r, \tag{8.3}$$

respectively. Using the continuity of $I_r(t)$ at time $t = \gamma_r$ and $t = \mu$, from (8.1)- (8.3), we have

$$S = W + \int_0^{\gamma_r} f(t)dt + \int_{\gamma_r}^{\mu} e^{\alpha_r(t-\gamma_r)^{\beta_r}} f(t)dt + f(\mu) \int_{\mu}^{t_r} e^{\alpha_r(t-\gamma_r)^{\beta_r}} dt. \tag{9}$$

The amount of the deteriorated items in RW is

$$\int_{\gamma_r}^{\mu} (e^{\alpha_r(t-\gamma_r)^{\beta_r}} - 1) f(t)dt + f(\mu) \int_{\mu}^{t_r} (e^{\alpha_r(t-\gamma_r)^{\beta_r}} - 1)dt. \tag{10}$$

Therefore,

$$C_{HR_{12}}(0, t_r) = c_{hr} \int_0^{t_r} e^{-rt} I_r(t)dt$$
$$= c_{hr} \left\{ \begin{array}{c} \int_0^{\gamma_r} e^{-rt} \left(S - W - \int_0^t f(v)dv\right) dt \\ + \int_{\gamma_r}^{\mu} e^{-rt} \left[S - W - \int_0^{\gamma_r} f(t)dt - \int_{\gamma_r}^{\mu} e^{\alpha_r(t-\gamma_r)^{\beta_r}} f(t)dt + e^{-\alpha_r(t-\gamma_r)^{\beta_r}} \int_t^{\mu} e^{\alpha_r(v-\gamma_r)^{\beta_r}} f(v)dv\right] dt \\ + f(\mu) \int_{\mu}^{t_r} e^{-rt} e^{-\alpha_r(t-\gamma_r)^{\beta_r}} \int_t^{t_r} e^{\alpha_r(v-\gamma_r)^{\beta_r}} dv\, dt \end{array} \right\}. \tag{11}$$

And

$$C_{DR_{12}}(\gamma_r, t_r)$$
$$= c_d \left[\int_{\gamma_r}^{\mu} e^{-rt} \left(e^{\alpha_r(t-\gamma_r)^{\beta_r}} - 1\right) f(t)dt + f(\mu) \int_{\mu}^{t_r} e^{-rt} \left(e^{\alpha_r(t-\gamma_r)^{\beta_r}} - 1\right) dt\right]. \tag{12}$$

Next, we discuss the model formulated in owned warehouse.

3.1.2 The model was formulated in own warehouse

Case 1. $\gamma_o \le t_r$, the deterioration starts before the inventory level of RW becomes zero. In this case, there is no change in $(0, \gamma_o)$, the inventory level is as follows.

$$I_o(t) = W, \qquad 0 \le t \le \gamma_o. \tag{13.1}$$

In (γ_o, t_r), the inventory decreases due to deterioration only; the differential equation is

$$\frac{dI_o(t)}{dt} = -Z_o(t)\, I_o(t), \qquad \gamma_o \le t \le t_r, \tag{13.2}$$

with the initial condition $I_o(\gamma_o) = W$. After time t_o, the inventory level depletes due to both deterioration and constant demand, the differential equation is

$$\frac{dI_o(t)}{dt} = -f(\mu) - Z_o(t)\, I_o(t), \qquad t_r \le t \le t_o \tag{13.3}$$

with the initial condition $I_o(t_o) = 0$. The solutions to (13.1)-(13.3) are as follows.

$$I_o(t) = We^{-\alpha_o(t-\gamma_o)^{\beta_o}}, \qquad \gamma_o \le t \le t_r, \tag{14.1}$$

and

$$I_o(t) = f(\mu)e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_t^{t_o} e^{\alpha_o(v-\gamma_o)^{\beta_o}} dv, \qquad t_r \le t \le t_o, \tag{14.2}$$

respectively. Using the continuity of $I_o(t)$ at time $t = t_r$, from (14.1) and (14.2), we have

$$W = f(\mu) \int_{t_r}^{t_o} e^{\alpha_o(t-\gamma_o)^{\beta_o}} dt \tag{15}$$

The amount of the deteriorated items in OW is

$$f(\mu) \int_{t_r}^{t_o} \left(e^{\alpha_o(v-\gamma_o)^{\beta_o}} - 1\right) dv \ . \tag{16}$$

Thus, the cumulative inventory in OW during $(0, t_o)$ is $\int_0^{t_o} I_o\,(t)dt$. Therefore,

$$C_{HO_{11}}(0, t_o) = c_{ho} \int_0^{t_o} e^{-rt} I_o(t)dt$$

$$= c_{ho} \begin{bmatrix} \int_0^{\gamma_o} W\, e^{-rt}dt + \int_{\gamma_o}^{t_r} We^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}\, dt \\ + \ f(\mu) \int_{t_r}^{t_o} e^{-r\,t}e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \ \int_t^{t_o} e^{\alpha_o(v-\gamma_o)^{\beta_o}} dv\, dt \end{bmatrix}. \tag{17}$$

And

$$C_{DO_{11}}(\gamma_o, t_o) = c_d f(\mu) \int_{t_r}^{t_o} e^{-rt} \left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) dt. \tag{18}$$

Case 2. $t_r \le \gamma_o < t_o$, the deterioration starts after the inventory level of RW becomes zero. In this case, there is no change in $(0, t_r)$, the inventory level is as follows.

$$I_o(t) = W, \qquad 0 \le t \le t_r. \tag{19.1}$$

In (t_r, γ_o), the inventory decreases due to constant demand only, the differential equation is

$$\frac{dI_o(t)}{dt} = -f(\mu), \qquad t_r \le t \le \gamma_o \ , \tag{19.2}$$

with the initial condition $I_o(t_r) = W$. After time γ_o, the inventory level depletes due to both deterioration and constant demand, the differential equation is

$$\frac{dI_o(t)}{dt} = -f(\mu) - Z_o(t)\, I_o(t), \qquad \gamma_o \le t \le t_o, \tag{19.3}$$

with the initial condition $I_o(t_o) = 0$. The solutions to (19.1)-(19.3) are as follows.

$$I_o(t) = W - f(\mu)(t - t_r), \qquad t_r \le t \le \gamma_o, \tag{20.1}$$

and

$$I_o(t) = f(\mu)e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_t^{t_o} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} d\nu, \qquad \gamma_o \leq t \leq t_o, \tag{20.2}$$

respectively. Using the continuity of $I_o(t)$ at time $t = \gamma_o$, from (20.1) and (20.2), we have

$$W = f(\mu)\left[(\gamma_o - t_r) + \int_{\gamma_o}^{t_o} e^{\alpha_o(t-\gamma_o)^{\beta_o}} dt\right]. \tag{21}$$

The amount of the deteriorated items in OW is

$$f(\mu)\int_{\gamma_o}^{t_o}\left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) dt. \tag{22}$$

Therefore,

$$\begin{aligned} C_{HO_{12}}(0,t_o) &= c_{ho}\int_0^{t_o} e^{-rt} I_o(t)dt \\ &= c_{ho}\begin{bmatrix} \int_0^{t_r} We^{-rt}dt + \int_{t_r}^{\gamma_o} e^{-rt}[W - f(\mu)(t-t_r)]dt \\ +f(\mu)\int_{\gamma_o}^{t_o} e^{-r\,t} e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_t^{t_o} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} d\nu\, dt \end{bmatrix}. \end{aligned} \tag{23}$$

And

$$C_{DO_{12}}(\gamma_o, t_o) = c_d f(\mu)\int_{\gamma_o}^{t_o} e^{-rt}\left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) dt. \tag{24}$$

By the time t_o, both warehouses are empty, and after that, shortages are allowed to occur. The partially backordered quantity is supplied to customers at the beginning of the next cycle. By the time T, the replenishment cycle restarts. During the interval (t_o, T), the backlogged level $B(t)$ at time t in OW is governed by the following differential equation:

$$\frac{dB(t)}{dt} = \delta(T-t)f(\mu), \qquad t_o \leq t \leq T, \tag{25}$$

with the boundary condition $B(t_o) = 0$. The solution to (25) is

$$B(t) = f(\mu)\int_{t_o}^{t} \delta(T-\nu)d\nu, \qquad t_o \leq t \leq T. \tag{26}$$

The number of lost sales at time t is

$$L(t) = f(\mu)\int_{t_o}^{t}[1 - \delta(T-\nu)]d\nu, \qquad t_o \leq t \leq T. \tag{27}$$

Thus,

$$\begin{aligned} C_B(t_o,T) &= c_b f(\mu)\int_{t_o}^{T} e^{-rt}\int_{t_o}^{t}\delta(T-\nu)d\nu\ dt \\ &= \frac{c_b}{r} f(\mu)\int_{t_o}^{T}(e^{-rt} - e^{-rT})\ \delta(T-t)d, \end{aligned} \tag{28}$$

and

$$C_L(t_o,T) = c_l f(\mu)\int_{t_o}^{T} e^{-rt}[1 - \delta(T-t)]dt, \tag{29}$$

respectively. In summary, the four cases of the present value of the total relevant cost per unit time for the research model during the cycle $[0,T]$ are given by

Total relevant cost = ordering cost + holding cost in RW + holding cost in OW + deteriorated items cost in RW+ deteriorated items cost in OW + backlogging cost + lost sales cost. *i.e.*,

$$TC_{1_1}(t_r,t_o,T)=[c_o+C_{HR_{11}}(0,t_r)+C_{HO_{11}}(0,t_o)+C_{DR_{11}}(\gamma_r,t_r)+C_{DO_{11}}(\gamma_o,t_o)$$
$$+C_B(t_o,T)+C_L(t_o,T)]/T, \quad \text{if} \quad min(\mu,\gamma_r)=\mu \text{ and } min(\gamma_o,t_r)=\gamma_o, \tag{30a}$$

$$TC_{12}(t_r,t_o,T)=[c_o+C_{HR_{11}}(0,t_r)+C_{HO_{12}}(0,t_o)+C_{DR_{11}}(\gamma_r,t_r)+C_{DO_{12}}(\gamma_o,t_o)$$
$$+C_B(t_o,T)+C_L(t_o,T)]/T, \quad \text{if} \quad min(\mu,\gamma_r)=\mu \text{ and } min(\gamma_o,t_r)=t_r, \tag{30b}$$

$$TC_{13}(t_r,t_o,T)=[c_o+C_{HR_{12}}(0,t_r)+C_{HO_{11}}(0,t_o)+C_{DR_{12}}(\gamma_r,t_r)+C_{DO_{11}}(\gamma_o,t_o)$$
$$+C_B(t_o,T)+C_L(t_o,T)]/T, \quad \text{if} \quad min(\mu,\gamma_r)=\gamma_r \text{ and } min(\gamma_o,t_r)=\gamma_o, \tag{30c}$$

$$TC_{14}(t_r,t_o,T)=[c_o+C_{HR_{12}}(0,t_r)+C_{HO_{12}}(0,t_o)+C_{DR_{12}}(\gamma_r,t_r)+C_{DO_{12}}(\gamma_o,t_o)$$
$$+C_B(t_o,T)+C_L(t_o,T)]/T, \quad \text{if} \quad min(\mu,\gamma_r)=\gamma_r \text{ and } min(\gamma_o,t_r)=t_r, \tag{30d}$$

3.2 Part 2: $\mu \geq t_r$

In RW, during $(0,t_r)$, there is only one case to be discussed. However, in OW, during the $(0,t_o)$, depending on the relation among γ_o, t_r, and μ, there are three cases to be considered, (i) $\gamma_o \leq t_r$ (ii) $t_r \leq \mu \leq \gamma_o$ and (iii) $t_r \leq \gamma_o \leq \mu$. Similarly, we also first discuss the model formulated in rented warehouse.

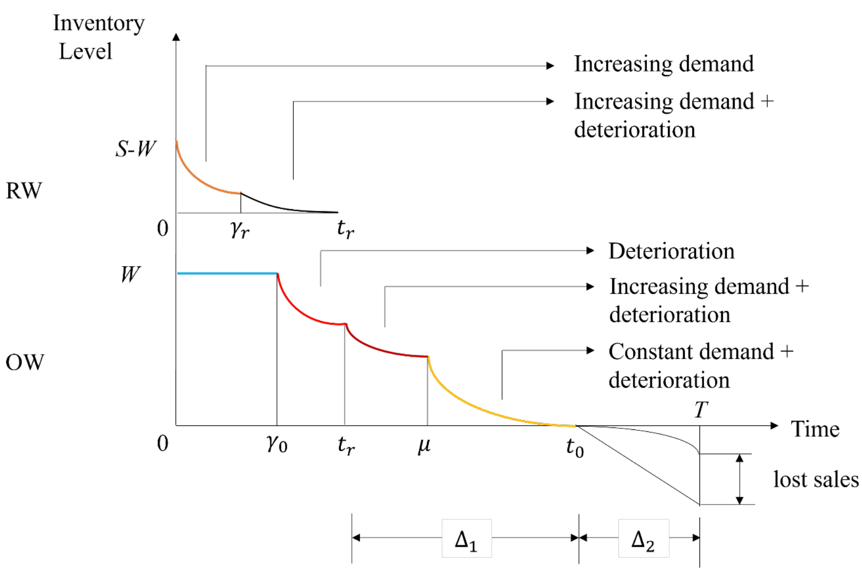

Fig 6. *Graphical Representation of The Research Model (where $\mu \geq t_r$ and $\gamma_0 \leq t_r$)*

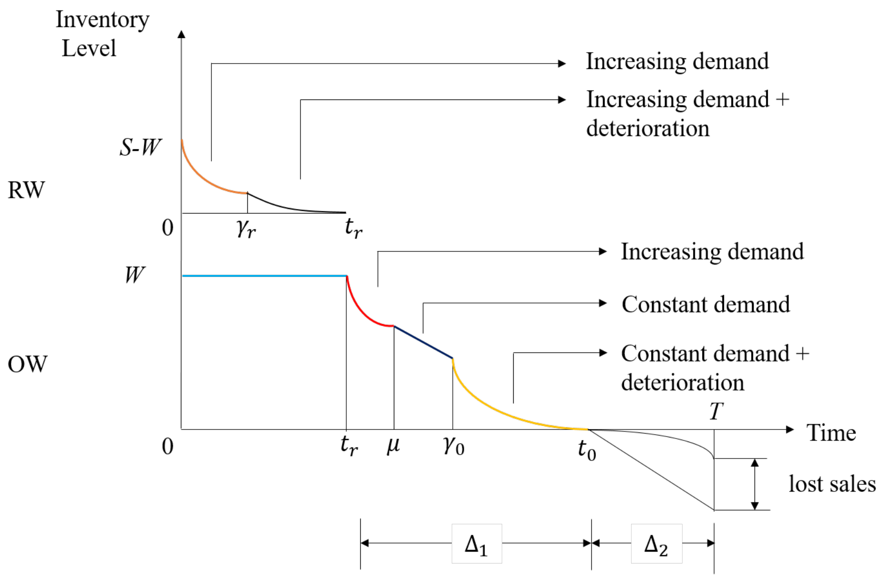

Fig 7. *Graphical Representation of The Research Model (where* $t_r \leq \mu \leq \gamma_0$*)*

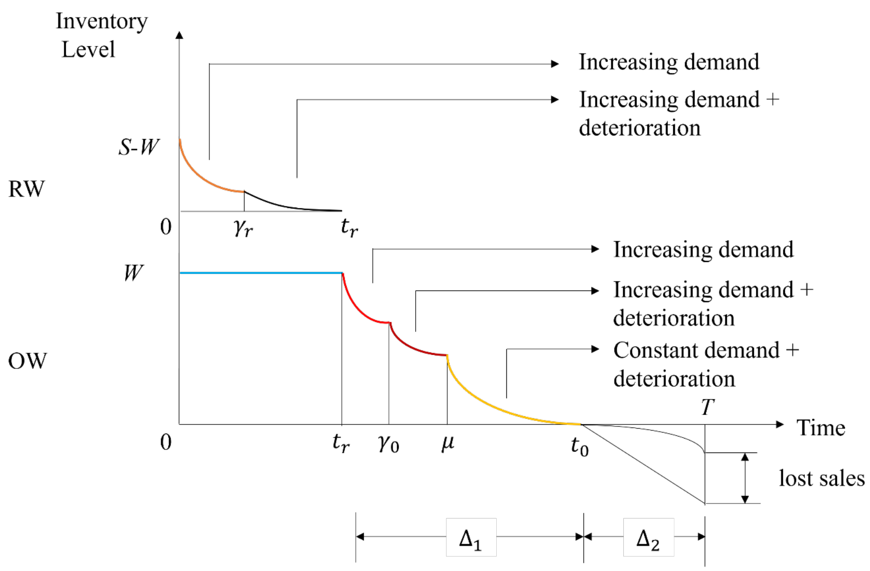

Fig 8. *Graphical Representation of The Research Model (where* $t_r \leq \gamma_0 \leq \mu$*)*

3.2.1. The model was formulated in rented warehouse.

In this case, the demand rate in RW is increasing with time. During $(0, \gamma_r)$, there is no deterioration, and the inventory gradually decreases due to demand, where $0 < t \leq \gamma_r$.

The inventory level at time t is governed by the following differential equation:

$$\frac{dI_r(t)}{dt} = -f(t), \qquad 0 \leq t \leq \gamma_r, \qquad (31.1)$$

with the boundary condition $I_r(0) = S - W$. During (γ_r, t_r), the inventory depletes due to both demand and deterioration, and it vanishes at $t = t_r$, it satisfies the following differential equation:

$$\frac{dI_r(t)}{dt} = -f(t) - Z_r(t)\, I_r(t), \qquad \gamma_r \le t \le t_r, \tag{31.2}$$

with the boundary condition $I_r(t_r) = 0$. The solutions to (31.1) and (31.2) are as follows.

$$I_r(t) = S - W - \int_0^t f(v)dv, \qquad 0 \le t \le \gamma_r, \tag{32.1}$$

and

$$I_r(t) = e^{-\alpha_r(t-\gamma_r)^{\beta_r}} \int_t^{t_r} e^{\alpha_r(v-\gamma_r)^{\beta_r}} f(v)dv, \qquad \gamma_r \le t \le t_r, \tag{32.2}$$

respectively. Using the continuity of $I_r(t)$ at time $t = \gamma_r$, from (32.1) and (32.2), we have

$$S = W + \int_0^{\gamma_r} f(t)dt + \int_{\gamma_r}^{t_r} e^{\alpha_r(t-\gamma_r)^{\beta_r}} f(t)dt. \tag{33}$$

The amount of the deteriorated items in RW is

$$\int_{\gamma_r}^{t_r} (e^{\alpha_r(t-\gamma_r)^{\beta_r}} - 1) f(t)dt. \tag{34}$$

Therefore,

$$C_{HR}(0, t_r) = c_{hr} \int_0^{t_r} e^{-rt} I_r(t)dt$$

$$= c_{hr} \begin{bmatrix} \int_0^{\gamma_r} e^{-rt}\left(S - W - \int_0^t f(v)dv\right)dt \\ + \int_{\gamma_r}^{t_r} e^{-rt} e^{-\alpha_r(t-\gamma_r)^{\beta_r}} \int_t^{t_r} e^{\alpha_r(v-\gamma_r)^{\beta_r}} f(v)dv\, dt \end{bmatrix}. \tag{35}$$

And

$$C_{DR}(\gamma_r, t_r) = c_d \int_{\gamma_r}^{t_r} e^{-rt}\left(e^{\alpha_r(t-\gamma_r)^{\beta_r}} - 1\right) f(t)dt. \tag{36}$$

3.2.2 The model was formulated in own warehouse.

In OW, during $(0, t_o)$, depending on the relation between γ_o and t_r, there are three cases to be considered, (i) $\gamma_o \le t_r$, (ii) $t_r \le \mu \le \gamma_o$ and (iii) $t_r \le \gamma_o \le \mu$.

Case 1. $\gamma_o \le t_r$, the deterioration starts before the inventory level of RW becomes zero. In this case, there is no change in $(0, \gamma_o)$, the inventory level is as follows.

$$I_o(t) = W, \qquad 0 \le t \le \gamma_o. \tag{37.1}$$

In (γ_o, t_r), the inventory decreases due to deterioration only, the differential equation is

$$\frac{dI_o(t)}{dt} = -Z_o(t)\, I_o(t), \qquad \gamma_o \le t \le t_r, \tag{37.2}$$

with the initial condition $I_o(\gamma_o) = W$. After time t_r, the inventory level depletes due to both deterioration and demand, the differential equation is

$$\frac{dI_o(t)}{dt} = -f(t) - Z_o(t)\, I_o(t), \qquad t_r \le t \le \mu, \tag{37.3}$$

$$\frac{dI_o(t)}{dt} = -f(\mu) - Z_o(t)\, I_o(t), \qquad \mu \le t \le t_o, \tag{37.4}$$

with the initial condition $I_o(t_o) = 0$. The solutions to (37.2)-(37.4) are as follows.

$$I_o(t) = We^{-\alpha_o(t-\gamma_o)^{\beta_o}}, \qquad \gamma_o \le t \le t_r, \tag{38.1}$$

$$I_o(t) = e^{-\alpha_o(t_r-\gamma_o)^{\beta_o}}\left(W - \int_{t_r}^{\mu} e^{\alpha_o(t-\gamma_o)^{\beta_o}} f(t)dt\right) +$$

$$e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_t^{\mu} e^{\alpha_o(v-\gamma_o)^{\beta_o}} f(v)dv, \qquad t_r \le t \le \mu, \tag{38.2}$$

$$I_o(t) = f(\mu)e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_t^{t_o} e^{\alpha_o(v-\gamma_o)^{\beta_o}} dv, \qquad \mu \le t \le t_o, \tag{38.3}$$

respectively. Using the continuity of $I_o(t)$ at time $t = \mu$, from (38.2) and (38.3), we have

$$W = \int_{t_r}^{\mu} e^{\alpha_o(t-\gamma_o)^{\beta_o}} f(t)dt + f(\mu)e^{\alpha_o(t_r-\gamma_o)^{\beta_o}} e^{-\alpha_o(\mu-\gamma_o)^{\beta_o}} \int_{\mu}^{t_o} e^{\alpha_o(t-\gamma_o)^{\beta_o}} dt. \tag{39}$$

The amounts of the deteriorated items in OW is

$$\int_{t_r}^{\mu} \left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) f(t)dt + f(\mu) \int_{\mu}^{t_o} \left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) dt. \tag{40}$$

Therefore,

$$C_{HO_1}(0, t_o) = c_{ho} \int_0^{t_o} e^{-rt} I_o(t)dt =$$

$$c_{ho}\left\{\begin{array}{c} \int_0^{\gamma_o} We^{-rt}dt + \int_{\gamma_o}^{t_r} We^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}\, dt \\ + \int_{t_r}^{\mu} e^{-rt}\left[e^{-\alpha_o(t_r-\gamma_o)^{\beta_o}}\left(W - \int_{t_r}^{\mu} e^{\alpha_o(t-\gamma_o)^{\beta_o}} f(t)dt\right) + e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_t^{\mu} e^{\alpha_o(v-\gamma_o)^{\beta_o}} f(v)dv\right] dt \\ + f(\mu) \int_{\mu}^{t_o} e^{-r\,t} e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_t^{t_o} e^{\alpha_o(v-\gamma_o)^{\beta_o}} dv\, dt \end{array}\right\}. \tag{41}$$

And

$$C_{DO_1}(\gamma_o, t_o)$$
$$= c_d\left[\int_{t_r}^{\mu} e^{-rt}\left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) f(t)dt + f(\mu)\int_{\mu}^{t_o} e^{-rt}\left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) dt\right]. \tag{42}$$

Case 2. $t_r \le \mu \le \gamma_o$, the deterioration starts after the inventory level of RW becomes zero. In this case, there is no change in $(0, t_r)$; the inventory level is as follows.

$$I_o(t) = W, \qquad 0 \le t \le t_r. \tag{43}$$

In (t_r, μ), the inventory decreases due to demand only; the differential equation is

$$\frac{dI_o(t)}{dt} = -f(t), \qquad t_r \le t \le \mu\ , \tag{44.1}$$

with the initial condition $I_o(t_r) = W$. After the time μ, the inventory level depletes due to the constant demand only, the differential equation is

$$\frac{dI_o(t)}{dt} = -f(\mu), \qquad \mu \le t \le \gamma_o\ , \tag{44.2}$$

and after the time γ_o, the inventory level depletes due to both deterioration and constant demand, the differential equations are

$$\frac{dI_o(t)}{dt} = -f(\mu) - Z_o(t)\, I_o(t), \qquad \gamma_o \le t \le t_o, \tag{44.3}$$

with the initial condition $I_o(t_o) = 0$. The solutions to (44.1) - (44.3) are as follows.

$$I_o(t) = W - \int_{t_r}^{t} f(\nu)d\nu, \qquad t_r \le t \le \mu, \tag{45.1}$$

$$I_o(t) = W - \int_{t_r}^{\mu} f(t)dt - f(\mu)(t-\mu), \qquad \mu \le t \le \gamma_o, \tag{45.2}$$

and

$$I_o(t) = f(\mu)e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_{t}^{t_o} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} d\nu, \qquad \gamma_o \le t \le t_o, \tag{45.3}$$

respectively. Using the continuity of $I_o(t)$ at time $t = \gamma_o$, from (45.2) and (45.3), we have

$$W = \int_{t_r}^{\mu} f(t)dt + f(\mu)\left[(\gamma_o - \mu) + \int_{\gamma_o}^{t_o} e^{\alpha_o(t-\gamma_o)^{\beta_o}} dt\right]. \tag{46}$$

The amounts of deteriorated items in OW is

$$f(\mu)\int_{\gamma_o}^{t_o} (e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1)dt. \tag{47}$$

Therefore,

$$C_{HO_2}(0,t_o) = c_{ho}\int_0^{t_o} e^{-rt} I_o(t)dt =$$

$$c_{ho}\begin{bmatrix} \int_0^{t_r} We^{-rt}\,dt + \int_{t_r}^{\mu} e^{-rt}\left(W - \int_{t_r}^{t} f(\nu)\,d\nu\right)dt \\ + \int_{\mu}^{\gamma_o} e^{-rt}\left(W - \int_{t_r}^{\mu} f(t)dt - f(\mu)(t-\mu)\right)dt \\ + f(\mu)\int_{\gamma_o}^{t_o} e^{-r\,t} e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_{t}^{t_o} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} d\nu\, dt \end{bmatrix}. \tag{48}$$

And

$$C_{DO_2}(t_r, t_o) = c_d f(\mu)\int_{\gamma_o}^{t_o} e^{-rt}\left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right)dt. \tag{49}$$

Case 3. $t_r \le \gamma_o \le \mu$

In (t_r, γ_o), the inventory decreases due to demand only; the differential equation is

$$\frac{dI_o(t)}{dt} = -f(t), \qquad t_r \le t \le \gamma_o, \tag{50.1}$$

with the initial condition $I_o(t_r) = W$. After the time γ_o, the inventory level depletes due to both deterioration and demand, the differential equation is

$$\frac{dI_o(t)}{dt} = -f(t) - Z_o(t)\, I_o(t), \qquad \gamma_o \le t \le \mu, \tag{50.2}$$

And after the time μ, the inventory level depletes due to both deterioration and constant demand, the differential equation is

$$\frac{dI_o(t)}{dt} = -f(\mu) - Z_o(t)\, I_o(t), \qquad \mu \le t \le t_o, \tag{50.3}$$

with the initial condition $I_o(t_o) = 0$. The solutions to (50.1)-(50.3) are as follows.

$$I_o(t) = W - \int_{t_r}^{t} f(\nu)d\nu, \qquad t_r \le t \le \gamma_o, \tag{51.1}$$

$$I_o(t) = W - \int_{t_r}^{\gamma_o} f(\nu)d\nu - \int_{\gamma_o}^{\mu} e^{\alpha_o(t-\gamma_o)^{\beta_o}} f(t)dt + e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_{t}^{\mu} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} f(\nu)d\nu, \qquad \gamma_o \le t \le \mu, \tag{51.2}$$

and

$$I_o(t) = f(\mu) e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_{t}^{t_o} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} d\nu, \qquad \mu \le t \le t_o, \tag{51.3}$$

respectively. Using the continuity of $I_o(t)$ at time $t = \mu$, from (51.2) and (51.3), we have

$$W = \int_{t_r}^{\gamma_o} f(t)dt + \int_{\gamma_o}^{\mu} e^{\alpha_o(t-\gamma_o)^{\beta_o}} f(t)dt + f(\mu) e^{-\alpha_o(\mu-\gamma_o)^{\beta_o}} \int_{\mu}^{t_o} e^{\alpha_o(t-\gamma_o)^{\beta_o}} dt. \tag{52}$$

The amounts of deteriorated items in OW is

$$\int_{\gamma_o}^{\mu} \left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) f(t)dt + f(\mu) \int_{\mu}^{t_o} \left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) dt. \tag{53}$$

Therefore,

$$C_{HO_3}(0, t_o) = c_{ho} \int_0^{t_o} e^{-rt} I_o(t)dt$$

$$= c_{ho} \left[\begin{array}{c} \int_0^{t_r} W e^{-rt} dt + \int_{t_r}^{\gamma_o} e^{-rt} \left(W - \int_{t_r}^{t} f(\nu)\, d\nu \right) dt \\ + \int_{\gamma_o}^{\mu} e^{-rt} \left(W - \int_{t_r}^{\gamma_o} f(\nu) d\nu - \int_{\gamma_o}^{\mu} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} f(t)dt + e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_{t}^{\mu} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} f(\nu)d\nu \right) dt \\ + f(\mu) \int_{\mu}^{t_o} e^{-r\,t} e^{-\alpha_o(t-\gamma_o)^{\beta_o}} \int_{t}^{t_o} e^{\alpha_o(\nu-\gamma_o)^{\beta_o}} d\nu\, dt \end{array} \right]. \tag{54}$$

And

$$C_{DO_3}(t_r, t_o) = c_d \left[\int_{\gamma_o}^{\mu} e^{-rt} \left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) f(t)dt + f(\mu) \int_{\mu}^{t_o} e^{-rt} \left(e^{\alpha_o(t-\gamma_o)^{\beta_o}} - 1\right) dt \right]. \tag{55}$$

Consequently, the following are the three cases of the present value of the total relevant cost per unit time for the research model during the cycle $[0, T]$.

$$TC_{2i}(t_r, t_o, T) = [c_o + C_{HR}(0, t_r) + C_{HO_i}(0, t_o) + C_{DR}(\gamma_r, t_r) + C_{DO_i}(\gamma_o, t_o) + C_B(t_o, T) + C_L(t_o, T)]/T, \quad \text{for } i = 1, 2, 3, \tag{56a-56c}$$

where $C_B(t_o, T)$ and $C_L(t_o, T)$ are the same as Equations (28) and (29). The objective of all the proposed models is to determine the time points t_r, t_o, T so that the total relevant cost per unit time of the inventory system is minimized.

4. Solutions to the research models

4.1 Part 1: $\mu \le t_r$

From (15) and (21), we know that t_0 is a function of t_r. Thus, $TC_{1i}(t_r, t_o, T)$ can be reduced to be a function of t_r and Δ_2, denoted by $TC_{1i}(t_r, \Delta_2)$, where $\Delta_2 = T - t_o$ and $\Delta_1 = t_o - t_r$, i =1, 2, 3, 4.

4.1.1 The case of $\mu \le \gamma_r$

The necessary conditions for $TC_{1i}(t_r, \Delta_2)$ in (30a) and (30b) to be minimized can be written as follows, for i = 1, 2.

$$\frac{\partial TC_{11}}{\partial t_r}(t_r + \Delta_1 + \Delta_2)$$

$$= f(\mu)\begin{bmatrix} c_{hr} e^{\alpha_r (t_r-\gamma_r)^{\beta_r}} \int_{\gamma_r}^{t_r} e^{-rt} e^{-\alpha_r (t-\gamma_r)^{\beta_r}} dt \\ + c_d e^{-rt_r}\left(e^{\alpha_r (t_r-\gamma_r)^{\beta_r}} - 1\right) - c_d e^{-rt_r}\left(e^{\alpha_o (t_r-\gamma_o)^{\beta_o}} - 1\right) \end{bmatrix}$$

$$+\left(1+\frac{d\Delta_1}{dt_r}\right) f(\mu) \begin{Bmatrix} \begin{bmatrix} c_{ho} e^{\alpha_o (t_o-\gamma_o)^{\beta_o}} \int_{t_r}^{t_o} e^{-rt} e^{-\alpha_o (t-\gamma_o)^{\beta_o}} dt \\ + c_d e^{-rt_o}\left(e^{\alpha_o (t_o-\gamma_o)^{\beta_o}} - 1\right) \end{bmatrix} \\ -e^{-rt_o}\left[\frac{c_b}{r}\left(1-e^{-r(T-t_o)}\right)\delta(T-t_o) + c_l\left(1-\delta(T-t_o)\right)\right] \\ +e^{-rT}\int_{t_o}^{T}\left[c_b\delta(T-t) + \left(\frac{c_b}{r}\left(e^{r(T-t)}-1\right) - c_l e^{r(T-t)}\right)\delta'(T-t)\right]dt \end{Bmatrix}$$

$$-TC_{11}\left(1+\frac{d\Delta_1}{dt_r}\right) = 0, \tag{57a}$$

$$\frac{\partial TC_{12}}{\partial t_r}(t_r + \Delta_1 + \Delta_2)$$

$$= f(\mu)\begin{bmatrix} c_{hr} e^{\alpha_r (t_r-\gamma_r)^{\beta_r}} \int_{\gamma_r}^{t_r} e^{-rt} e^{-\alpha_r (t-\gamma_r)^{\beta_r}} dt \\ + c_d e^{-rt_r}\left(e^{\alpha_r (t_r-\gamma_r)^{\beta_r}} - 1\right) + \frac{c_{ho}}{r}\left(e^{-rt_r} - e^{-r\gamma_o}\right) \end{bmatrix}$$

$$+\left(1+\frac{d\Delta_1}{dt_r}\right) f(\mu) \begin{Bmatrix} \begin{bmatrix} c_{ho} e^{\alpha_o (t_o-\gamma_o)^{\beta_o}} \int_{t_r}^{t_o} e^{-rt} e^{-\alpha_o (t-\gamma_o)^{\beta_o}} dt \\ + c_d e^{-rt_o}\left(e^{\alpha_o (t_o-\gamma_o)^{\beta_o}} - 1\right) \end{bmatrix} \\ -e^{-rt_o}\left[\frac{c_b}{r}\left(1-e^{-r(T-t_o)}\right)\delta(T-t_o) + c_l\left(1-\delta(T-t_o)\right)\right] \\ +e^{-rT}\int_{t_o}^{T}\left[c_b\delta(T-t) + \left(\frac{c_b}{r}\left(e^{r(T-t)}-1\right) - c_l e^{r(T-t)}\right)\delta'(T-t)\right]dt \end{Bmatrix}$$

$$-TC_{12}(1+\frac{d\Delta_1}{dt_r}) = 0, \tag{57b}$$

and

$$\frac{\partial TC_{1i}}{\partial \Delta_2}(t_r+\Delta_1+\Delta_2)$$
$$= e^{-rT}f(\mu)\int_{t_o}^{T}\left[c_b\delta(T-t)+\left(\frac{c_b}{r}(e^{r(T-t)}-1)-c_le^{r(T-t)}\right)\delta'(T-t)\right]dt-TC_{1i}$$
$$=0, \qquad i=1,\ 2,\ 3,\ 4. \tag{58}$$

From (15) and (21), we have

$$1+\frac{d\Delta_1}{dt_r}=e^{-\alpha_o[(t_o-\gamma_o)^{\beta_o}-(t_r-\gamma_o)^{\beta_o}]}, \qquad \text{if } \gamma_o\le t_r, \tag{59a}$$

$$1+\frac{d\Delta_1}{dt_r}=e^{-\alpha_o(t_o-\gamma_o)^{\beta_o}} \qquad \text{if } \gamma_o\ge t_r. \tag{59b}$$

Thus, from (57a), (58) and (59a), (57b), (58) and (59b), the following results can be obtained.

$$\frac{c_b}{r}[(1-e^{-r\Delta_2})\delta(\Delta_2)+c_l(1-\delta(\Delta_2))]$$

$$=e^{\alpha_o[(t_o-\gamma_o)^{\beta_o}-(t_r-\gamma_o)^{\beta_o}]}e^{rt_o}\begin{bmatrix} c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\gamma_r}^{t_r}e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt \\ +c_de^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right)-c_de^{-rt_r}\left(e^{\alpha_o(t_r-\gamma_o)^{\beta_o}}-1\right)\end{bmatrix}$$
$$+e^{rt_o}\left[c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{t_r}^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt+c_de^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1)\right)\right],$$
$$\text{if } \gamma_o\le t_r, \tag{60a}$$

and

$$\frac{c_b}{r}[(1-e^{-r\Delta_2})\delta(\Delta_2)+c_l(1-\delta(\Delta_2))]$$

$$=e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}e^{rt_o}\begin{bmatrix} c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\gamma_r}^{t_r}e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt \\ +c_de^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right)+\frac{c_{ho}}{r}(e^{-rt_r}-e^{-r\gamma_o})\end{bmatrix}$$
$$+e^{rt_o}\left[c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{\gamma_o}^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt+c_de^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1)\right)\right],$$
$$\text{if } \gamma_o\le t_r, \tag{60b}$$

Thus, Δ_2 is also a function of t_r. Consequently, if t_r is known, then Δ_2 can be determined by (60a) if $min(\gamma_o, t_r) = \gamma_o$, and (60b) if $min(\gamma_o, t_r) = t_r$, respectively.

For notational convenience, let K_1 be the right-hand side of (60a) and (60b), then the following results can be obtained.

Theorem 1. If $K_1 > 0$ and $dK_1/dt_r > 0$, then the optimal replenishment schedule satisfies

(i) (57a) and (58), if $min(\gamma_o, t_r) = \gamma_o$, and

(ii)(57b) and (58), if $min(\gamma_o, t_r) = t_r$,

is uniquely determined and minimizes TC_{1i}, for $i = 1, 2$.

Proof: The proof is similar to Yang's (2006).

4.1.2 The case of $\mu \geq \gamma_r$

The necessary conditions for $TC_{1i}(t_r, \Delta_2)$ in (30c) and (30d) to be minimized can be written as follows, for $i = 3, 4$.

$$\frac{\partial TC_{13}}{\partial t_r}(t_r + \Delta_1 + \Delta_2)$$

$$= f(\mu)\begin{bmatrix} c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\mu}^{t_r} e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt \\ +c_d e^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right) - c_d e^{-rt_r}\left(e^{\alpha_o(t_r-\gamma_o)^{\beta_o}}-1\right)\end{bmatrix}$$

$$+\left(1+\frac{d\Delta_1}{dt_r}\right)f(\mu)\left\{\begin{matrix} \begin{bmatrix} c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{\gamma_o}^{t_o} e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt \\ +c_d e^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1\right)\end{bmatrix} \\ -e^{-rt_o}\left[\frac{c_b}{r}\left(1-e^{-r(T-t_o)}\right)\delta(T-t_o) + c_l\left(1-\delta(T-t_o)\right)\right] \\ +e^{-rT}\int_{t_o}^{T}\left[c_b\delta(T-t) + \left(\frac{c_b}{r}\left(e^{r(T-t)}-1\right) - c_l e^{r(T-t)}\right)\delta'(T-t)\right]dt \end{matrix}\right\}$$

$$-TC_{13}\left(1+\frac{d\Delta_1}{dt_r}\right) = 0, \tag{61a}$$

and

$$\frac{\partial TC_{14}}{\partial t_r}(t_r + \Delta_1 + \Delta_2)$$

$$= f(\mu)\begin{bmatrix} c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\mu}^{t_r} e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt \\ +c_d e^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right) + \frac{c_{ho}}{r}\left(e^{-rt_r}-e^{-r\gamma_o}\right)\end{bmatrix}$$

$$+\left(1+\frac{d\Delta_1}{dt_r}\right)f(\mu)\left\{\begin{array}{c}\left[\begin{array}{c}c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{\gamma_o}^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt\\+c_de^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1\right)\end{array}\right]\\-e^{-rt_o}\left[\frac{c_b}{r}\left(1-e^{-r(T-t_o)}\right)\delta(T-t_o)+c_l\left(1-\delta(T-t_o)\right)\right]\\+e^{-rT}\int_{t_o}^{T}\left[c_b\delta(T-t)+\left(\frac{c_b}{r}\left(e^{r(T-t)}-1\right)-c_le^{r(T-t)}\right)\delta'(T-t)\right]dt\end{array}\right\}$$

$$-TC_{14}\left(1+\frac{d\Delta_1}{dt_r}\right)=0, \tag{61b}$$

Thus, from (61a), (58) and (59a), (61b), (58) and (59b), the following results can be obtained.

$$\frac{c_b}{r}\left[(1-e^{-r\Delta_2})\delta(\Delta_2)+c_l(1-\delta(\Delta_2))\right]$$

$$=e^{\alpha_o[(t_o-\gamma_o)^{\beta_o}-(t_r-\gamma_o)^{\beta_o}]}e^{rt_o}\left[\begin{array}{c}c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\mu}^{t_r}e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt\\+c_de^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right)-c_de^{-rt_r}\left(e^{\alpha_o(t_r-\gamma_o)^{\beta_o}}-1\right)\end{array}\right]$$

$$+e^{rt_o}\left[c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{t_r}^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt+c_de^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1\right)\right],$$

$$\text{if } \gamma_o\le t_r, \tag{62a}$$

and

$$\frac{c_b}{r}\left[(1-e^{-r\Delta_2})\delta(\Delta_2)+c_l(1-\delta(\Delta_2))\right]$$

$$=e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}e^{rt_o}\left[\begin{array}{c}c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\mu}^{t_r}e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt\\+c_de^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right)+\frac{c_{ho}}{r}(e^{-rt_r}-e^{-r\gamma_o})\end{array}\right]$$

$$+e^{rt_o}\left[c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{\gamma_o}^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt+c_de^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1\right)\right],$$

$$\text{if } \gamma_o\ge t_r. \tag{62b}$$

Thus, Δ_2 is also a function of t_r. Consequently, if t_r is known, then Δ_2 can be determined by (62a) if $min(\gamma_o,t_r)=\gamma_o$, and (62b) if $min(\gamma_o,t_r)=t_r$, respectively.

Similarly, for notational convenience, let K_2 be the right-hand side of (62a) and (62b), then the following results can be obtained.

Theorem 2. If $K_2>0$ and $dK_2/dt_r>0$, then the optimal replenishment schedule satisfies

(61a) and (58), if $min(\gamma_o, t_r) = \gamma_o$, and

(61b) and (58), if $min(\gamma_o, t_r) = t_r$,

is uniquely determined and minimizes TC_{1i}, for $i = 3, 4$.

Proof: The proof is similar to Yang's (2006).

Note that $TC_{1i}(t_r, t_o, T)$ is a continuous function on a compact set $[0, T]$; for $i = 1, 2, 3, 4$, we know a global minimum solution exists. It is clear that $TC_{1i}(t_r, t_o, T)$ is neither minimum at $t_r = 0$ nor at $t_r = T$. The optimal solution obtained from (57a, b) and (50), (61a, b) and (58) is not on the boundary; hence, the unique solution is a global optimal.

4.2 Part 2: $\mu \geq t_r$

From (39), (46) and (52), we know that t_0 is a function of t_r. Thus, $TC_{2i}(t_r, t_o, T)$ can be reduced to be a function of t_r and Δ_2, denoted by $TC_{2i}(t_r, \Delta_2)$, where $\Delta_2 = T - t_o$ and $\Delta_1 = t_o - t_r$, $i = 1, 2, 3$.

The necessary conditions for $TC_{2i}(t_r, \Delta_2)$ in (56a)-(56c) to be minimized can be written as follows, for $i = 1, 2, 3$.

$$\frac{\partial TC_{21}}{\partial t_r}(t_r + \Delta_1 + \Delta_2)$$

$$= f(t_r)\begin{bmatrix} c_{hr} e^{\alpha_r (t_r - \gamma_r)^{\beta_r}} \int_{\gamma_r}^{t_r} e^{-rt} e^{-\alpha_r (t-\gamma_r)^{\beta_r}} dt \\ + c_d e^{-rt_r}\left(e^{\alpha_r (t_r - \gamma_r)^{\beta_r}} - 1\right) - c_d e^{-rt_r}\left(e^{\alpha_o (t_r - \gamma_o)^{\beta_o}} - 1\right) \\ + \frac{c_{ho}}{r}\left(e^{-rt_r} - e^{-r\mu}\right) \end{bmatrix}$$

$$-c_{ho}\alpha_o\beta_o (t_r - \gamma_o)^{\beta_o - 1} e^{-\alpha_o (t_r - \gamma_o)^{\beta_o}} \left(W - \int_{t_r}^{\mu} e^{\alpha_o (t-\gamma_o)^{\beta_o}} f(t) dt\right) \int_{t_r}^{\mu} e^{-rt} dt$$

$$+\left(1 + \frac{d\Delta_1}{dt_r}\right) f(\mu) \left\{ \begin{matrix} \begin{bmatrix} c_{ho} e^{\alpha_o (t_o - \gamma_o)^{\beta_o}} \int_{\mu}^{t_o} e^{-rt} e^{-\alpha_o (t-\gamma_o)^{\beta_o}} dt \\ + c_d e^{-rt_o}\left(e^{\alpha_o (t_o - \gamma_o)^{\beta_o}} - 1\right) \end{bmatrix} \\ -e^{-rt_o}\left[\frac{c_b}{r}\left(1 - e^{-r(T-t_o)}\right)\delta(T - t_o) + c_l\left(1 - \delta(T - t_o)\right)\right] \\ + e^{-rT}\int_{t_o}^{T}\left[c_b \delta(T-t) + \left(\frac{c_b}{r}\left(e^{r(T-t)} - 1\right) - c_l e^{r(T-t)}\right)\delta'(T-t)\right] dt \end{matrix} \right\}$$

$$-TC_{21}\left(1 + \frac{d\Delta_1}{dt_r}\right) = 0, \tag{63a}$$

$$\frac{\partial TC_{22}}{\partial t_r}(t_r + \Delta_1 + \Delta_2)$$

$$= f(t_r)\begin{bmatrix} c_{hr} e^{\alpha_r (t_r - \gamma_r)^{\beta_r}} \int_{\gamma_r}^{t_r} e^{-rt} e^{-\alpha_r (t-\gamma_r)^{\beta_r}} dt \\ + c_d e^{-rt_r}\left(e^{\alpha_r (t_r - \gamma_r)^{\beta_r}} - 1\right) + \frac{c_{ho}}{r}\left(e^{-rt_r} - e^{-r\gamma_o}\right) \end{bmatrix}$$

$$+\left(1+\frac{d\Delta_1}{dt_r}\right)f(\mu)\left\{\begin{matrix}\begin{bmatrix}c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_\mu^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt\\+c_d e^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1\right)\end{bmatrix}\\-e^{-rt_o}\left[\frac{c_b}{r}\left(1-e^{-r(T-t_o)}\right)\delta(T-t_o)+c_l\left(1-\delta(T-t_o)\right)\right]\\+e^{-rT}\int_{t_o}^{T}\left[c_b\delta(T-t)+\left(\frac{c_b}{r}(e^{r(T-t)}-1)-c_l e^{r(T-t)}\right)\delta'(T-t)\right]dt\end{matrix}\right\}$$

$$-TC_{22}(1+\frac{d\Delta_1}{dt_r})\ =0, \tag{63b}$$

$$\frac{\partial TC_{23}}{\partial t_r}(t_r+\Delta_1+\Delta_2)$$

$$=f(t_r)\begin{bmatrix}c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\gamma_r}^{t_r}e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt\\+c_d e^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right)+\frac{c_{ho}}{r}(e^{-rt_r}-e^{-r\mu})\end{bmatrix}$$

$$+\left(1+\frac{d\Delta_1}{dt_r}\right)f(\mu)\left\{\begin{matrix}\begin{bmatrix}c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_\mu^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt\\+c_d e^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1\right)\end{bmatrix}\\-e^{-rt_o}\left[\frac{c_b}{r}\left(1-e^{-r(T-t_o)}\right)\delta(T-t_o)+c_l\left(1-\delta(T-t_o)\right)\right]\\+e^{-rT}\int_{t_o}^{T}\left[c_b\delta(T-t)+\left(\frac{c_b}{r}(e^{r(T-t)}-1)-c_l e^{r(T-t)}\right)\delta'(T-t)\right]dt\end{matrix}\right\}$$

$$-TC_{23}(1+\frac{d\Delta_1}{dt_r})\ =0, \tag{63c}$$

and

$$\frac{\partial TC_{2i}}{\partial \Delta_2}(t_r+\Delta_1+\Delta_2)$$

$$=e^{-rT}f(\mu)\int_T^{t_s}\left[c_b\delta(T-t)+\left(\frac{c_b}{r}\left(e^{r(T-t)}-1\right)-c_l e^{r(T-t)}\right)\delta'(T-t)\right]dt$$

$$-TC_{2i}=0, \qquad i=1,2,3. \tag{64}$$

From (39), (46) and (52), we have

$$1+\frac{d\Delta_1}{dt_r}=e^{-\alpha_o(t_o-\gamma_o)^{\beta_o}}\left[\frac{f(t_\gamma)e^{\alpha_o(\mu-\gamma_o)^{\beta_o}}}{f(\mu)}-\alpha_o\beta_o(t-\gamma_o)^{\beta_o-1}\int_\mu^{t_o}e^{\alpha_o(t-\gamma_o)^{\beta_o}}dt\right],$$

$$\text{if } \gamma_o\le t_r\le\mu. \tag{65a}$$

$$1+\frac{d\Delta_1}{dt_r}=f(t_\gamma)e^{-\alpha_o(t_o-\gamma_o)^{\beta_o}}/f(\mu), \qquad \text{if } t_r\le\mu\le\gamma_o. \tag{65b}$$

$$1+\frac{d\Delta_1}{dt_r}=f(t_\gamma)e^{-\alpha_o[(t_o-\gamma_o)^{\beta_o}-(\mu-\gamma_o)^{\beta_o}]}/f(\mu), \qquad \text{if } t_r\le\gamma_o\le\mu. \tag{65c}$$

Thus, from (63a), (64) and (65a); (63b), (64) and (65b); (63c), (64) and (65c), the following results can be obtained.

$$\frac{c_b}{r}[(1-e^{-r\Delta_2})\delta(\Delta_2)+c_l(1-\delta(\Delta_2))]\,f(\mu)$$
$$=e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\left[f(t_\gamma)e^{\alpha_o(\mu-\gamma_o)^{\beta_o}}-f(\mu)\alpha_o\beta_o(t-\gamma_o)^{\beta_o-1}\int_{\mu}^{t_o}e^{\alpha_o(t-\gamma_o)^{\beta_o}}dt\right]e^{rt_o}$$
$$\left\{\begin{array}{c} f(t_r)\left[\begin{array}{c} c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\gamma_r}^{t_r}e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt \\ +c_d e^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right)-c_d e^{-rt_r}\left(e^{\alpha_o(t_r-\gamma_o)^{\beta_o}}-1\right) \\ +\frac{c_{ho}}{r}(e^{-rt_r}-e^{-r\mu}) \end{array}\right] \\ -c_{ho}\alpha_o\beta_o(t_r-\gamma_o)^{\beta_o-1}e^{-\alpha_o(t_r-\gamma_o)^{\beta_o}}\left(W-\int_{t_r}^{\mu}e^{\alpha_o(t-\gamma_o)^{\beta_o}}f(t)dt\right)\int_{t_r}^{\mu}e^{-rt}dt \end{array}\right\}$$
$$+e^{rt_o}f(\mu)\left[c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{\mu}^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt+c_d e^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1)\right)\right],$$
$$\text{if } \gamma_o\le t_r\le\mu. \tag{66a}$$

$$\frac{c_b}{r}[(1-e^{-r\Delta_2})\delta(\Delta_2)+c_l(1-\delta(\Delta_2))]f(\mu)$$
$$=e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}e^{rt_o}f(t_r)\left[\begin{array}{c} c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\gamma_r}^{t_r}e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt \\ +c_d e^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right)+\frac{c_{ho}}{r}(e^{-rt_r}-e^{-r\gamma_o}) \end{array}\right]$$
$$+e^{rt_o}f(\mu)\left[c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{\gamma_o}^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt+c_d e^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1\right)\right],$$
$$\text{if } t_r\le\mu\le\gamma_o. \tag{66b}$$

$$\frac{c_b}{r}[(1-e^{-r\Delta_2})\delta(\Delta_2)+c_l(1-\delta(\Delta_2))]f(\mu)=$$
$$e^{\alpha_o[(t_o-\gamma_o)^{\beta_o}-(\mu-\gamma_o)^{\beta_o}]}e^{rt_o}f(t_r)\left[\begin{array}{c} c_{hr}e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}\int_{\gamma_r}^{t_r}e^{-rt}e^{-\alpha_r(t-\gamma_r)^{\beta_r}}dt \\ +c_d e^{-rt_r}\left(e^{\alpha_r(t_r-\gamma_r)^{\beta_r}}-1\right)+\frac{c_{ho}}{r}(e^{-rt_r}-e^{-r\mu}) \end{array}\right]$$
$$+e^{rt_o}f(\mu)\left[c_{ho}e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}\int_{\mu}^{t_o}e^{-rt}e^{-\alpha_o(t-\gamma_o)^{\beta_o}}dt+c_d e^{-rt_o}\left(e^{\alpha_o(t_o-\gamma_o)^{\beta_o}}-1\right)\right],$$
$$\text{if } t_r\le\gamma_o\le\mu. \tag{66c}$$

Thus, Δ_2 is also a function of t_r. Consequently, if t_r is known, then Δ_2 can be determined by (66a) if $\gamma_o\le t_r\le\mu$; (66b) if $t_r\le\mu\le\gamma_o$; and (66c) if $t_r\le\gamma_o\le\mu$.

Similarly, for notational convenience, let K_3 be the right-hand side of (66a)-(66c), then the following results can be obtained.

Theorem 3. If $K_3>0$ and $dK_3/dt_r>0$, then the optimal replenishment schedule satisfies

(i) (63a) and (64), if $\gamma_o \leq t_r \leq \mu$,

(ii) (63b) and (64), if $t_r \leq \mu \leq \gamma_o$, and

(iii) (63c) and (64), if $t_r \leq \gamma_o \leq \mu$,

is uniquely determined and minimizes TC_{2i}, for $i = 1, 2, 3$.

Proof: The proof is similar to Yang's (2006).

Note that $TC_{2i}(t_r, t_o, T)$ is a continuous function on a compact set $[0, T]$, for $i = 1, 2, 3$, we know a global minimum solution exists. It is clear that $TC_{2i}(t_r, t_o, T)$ is neither minimum at $t_r = 0$ nor at $t_r = T$. The optimal solution obtained from (63a)-(63c) and (64) is not on the boundary, and hence the unique solution is a global optimal.

In summary, from the relations among the parameters μ, r_r, r_o and $\boldsymbol{t_r}$, we have the following results in Table 2.

Table 2. *Each Proposed Model with Its Constraints*

Model	Constraint
TC_{11}	$\mu \leq r_r \leq r_o \leq \boldsymbol{t_r}$
TC_{12}	$\mu \leq \boldsymbol{t_r} \leq r_r \leq r_o$ $\mu \leq r_r \leq \boldsymbol{t_r} \leq r_o$
TC_{13}	$r_r \leq r_o \leq \mu \leq \boldsymbol{t_r}$ $r_r \leq \mu \leq r_o \leq \boldsymbol{t_r}$
TC_{14}	$r_r \leq \mu \leq \boldsymbol{t_r} \leq r_o$
TC_{21}	$r_r \leq r_o \leq \boldsymbol{t_r} \leq \mu$
TC_{22}	$r_r \leq \boldsymbol{t_r} \leq \mu \leq r_o$
TC_{23}	$r_r \leq \boldsymbol{t_r} \leq r_o \leq \mu$

Thus, according to Table 2 and the relation among μ, r_r, and r_o, the proposed model may be considered in the following Table 3.

Table 3. *Relation And Proposed Model May Be Considered*

Relation	The model may be considered
$\mu \leq r_r \leq r_o$	TC_{11}、TC_{12}
$r_r \leq \mu \leq r_o$	TC_{13}、TC_{14}、TC_{22}
$r_r \leq r_o \leq \mu$	TC_{13}、TC_{21}、TC_{23}

Based on the above-discussed equations, Theorems 1-3, and Table 3, an algorithm for finding the optimal solution is established (see Appendix A).

5. Numerical examples

Many products might exhibit ramp-type demand with a deterioration rate that can be modeled using a three-parameter Weibull distribution, such as fashion apparel, high-tech gadgets, perishable food products, or seasonal products like holiday decorates. These examples illustrate how ramp-type demand and Weibull distribution deterioration are relevant in various industries, each facing unique inventory and demand management challenges. The demand rate and parameters for Weibull deterioration distribution and relevant costs are stated in the following examples.

Example 1. Let $f(t) = 100 + 50t$, W =75, μ = 0.02, $\delta(t) = e^{-0.6t}$, c_o =150, c_{ho}= 0.5, c_{ho}= 0.8, c_b= 15, c_l= 30, c_d=7.5, α_o= 0.05, β_o= 2, γ_o= 0.05, α_r= 0.03, β_r= 2, γ_r= 0.03 and r = 0.06 in appropriate units. By Algorithm A and Table 3, we know that if $\mu \leq r_r \leq r_o$, then TC_{11} and TC_{12} are considered. Thus, we get the optimal present value of the total relevant cost per unit time $TC^* = min\{TC_{11,}TC_{12}\}$ = $min\{164.450, 167.185\}$ = 164.450 = TC_{11} , the optimal replenishment cycle T= 1.53229, the optimal stock period in OW t_o = 1.47693, the optimal stock period in RW t_r = 0.77641 and the optimal order quantity S = 153.827.

Example 2. Let $f(t) = 200 + 100t$, W =220, μ = 0.04, $\delta(t) = e^{-0.6t}$, c_o =150, c_{ho}= 0.5, c_{ho}= 0.8, c_b= 15, c_l= 30, c_d=7.5, α_o= 0.05, β_o= 2, γ_o= 0.05, α_r= 0.03, β_r= 2, γ_r= 0.03 and r = 0.06 in appropriate units. By Algorithm A and Table 3, we know that if $r_r \leq \mu \leq r_o$, then TC_{13}, TC_{14}, and TC_{22} are considered. Thus, we get the optimal present value of the total relevant cost per unit time $TC^* = min\{TC_{13,}TC_{14,}TC_{22}\}$ = $min\{214.316, 214.668, 214.819\}$ =214.316 = TC_{13} , the optimal replenishment cycle T = 1.19277, the optimal stock period in OW t_o = 1.15807, the optimal stock period in RW t_r = 0.10232 and the optimal order quantity S = 240.79.

Example 3. Let $f(t) = 150 + 300t$, W =150, μ =0.06, $\delta(t) = e^{-0.6t}$, c_o =150, c_{ho}= 0.5, c_{ho}= 0.8, c_b= 15, c_l= 30, c_d=7.5, α_o= 0.05, β_o= 2, γ_o= 0.05, α_r= 0.03, β_r= 2, γ_r= 0.03 and r = 0.06 in appropriate units. By Algorithm A and Table 3, we know that if $r_r \leq r_o \leq \mu$, then TC_{13}, TC_{21}, and TC_{23} are considered. Thus, we get the optimal present value of the total relevant cost per unit time $TC^* = min\{TC_{13,}\, TC_{21,}\, TC_{23}\}$= $\min\{199.751, 209.586, 210.379\}$ = 199.751= TC_{13} , the replenishment cycle T= 1.25340, the stock period in OW t_o=1.21391, the stock period in RW t_r= 0.34690 and the order quantity S = 207.79.

Example 4. This example employs the parameter values from Example 3 to assess whether input parameter variations affect the optimal solution's sensitivity. The numerical outcomes of the sensitivity analysis are displayed in Table 4.

Table 4. *Sensitivity Analysis for Each Parameter*

Parameter \ Decision	TC	t_o	T	t_r	S
$\mu = 0.02$	193.14	1.25455	1.29581	0.32182	200.18
$\mu = 0.04$	196.48	1.23366	1.27401	0.33500	204.08
$\mu = 0.06$	199.75	1.21391	1.25340	0.34690	207.79
Trend	↗	↘	↘	↗	↗
$b = 200$	196.47	1.23365	1.27400	0.33499	203.95
$b = 300$	199.75	1.21391	1.25340	0.34690	207.79
$b = 400$	202.98	1.19521	1.23390	0.35770	211.58
Trend	↗	↘	↘	↗	↗
$W = 100$	204.44	1.19871	1.23911	0.62557	204.91
$W = 150$	199.75	1.21391	1.25340	0.34690	207.79
$W = 200$	197.33	1.25408	1.29319	0.09271	215.04
Trend	↘	↗	↗	↘	↗
$c_o = 100$	156.90	1.04858	1.07916	0.17229	178.41
$c_o = 150$	199.75	1.21391	1.25340	0.34690	207.79
$c_o = 200$	237.40	1.35431	1.40182	0.49696	233.12
Trend	↗	↗	↗	↗	↗
$c_{ho} = 0.4$	190.91	1.23069	1.26844	0.36476	210.80
$c_{ho} = 0.5$	199.75	1.21391	1.25340	0.34690	207.79
$c_{ho} = 0.6$	208.52	1.19627	1.23748	0.32816	204.64
Trend	↗	↘	↘	↘	↘
$c_{hr} = 0.7$	198.91	1.23535	1.27472	0.36971	211.64
$c_{hr} = 0.8$	199.75	1.21391	1.25340	0.34690	207.79
$c_{hr} = 0.9$	200.51	1.19496	1.23456	0.32677	204.40
Trend	↗	↘	↘	↘	↘
$c_d = 5$	190.30	1.29724	1.33502	0.43577	222.78
$c_d = 7.5$	199.75	1.21391	1.25340	0.34690	207.79
$c_d = 10$	208.07	1.14969	1.19070	0.27881	196.33
Trend	↗	↘	↘	↘	↘
$\beta_o = 1$	203.57	1.39456	1.43527	0.54182	240.71
$\beta_o = 2$	199.75	1.21391	1.25340	0.34690	207.79
$\beta_o = 3$	160.24	1.60117	1.63348	0.69929	267.44
Trend	↘	N/A	N/A	N/A	N/A
$\alpha_o = 0.04$	194.31	1.25773	1.29623	0.38786	214.70
$\alpha_o = 0.05$	199.75	1.21391	1.25340	0.34690	207.79
$\alpha_o = 0.06$	204.82	1.17530	1.21572	0.31061	201.68
Trend	↗	↘	↘	↘	↘

From the Table 4, the following managerial insights are obtained.

(1) When the growth period μ or the slope of demand rate b increases, the optimal present value of total cost TC, the optimal stock period in RW t_r and the optimal order quantity S also increases, while the optimal stock period in OW t_o and the optimal replenishment cycle T decrease.

(2) As the capacity of the owned warehouse W grows, the optimal stock period in OW t_o, the optimal replenishment cycle T and the optimal order quantity S exhibit an upward trend,

whereas the optimal present value of total cost TC and the stock period in RW t_r experience a decline.

(3) When the replenishment cost c_o elevates, the optimal present value of total cost TC, the optimal stock period in OW t_o, the optimal replenishment cycle T, the optimal stock period in RW t_r, and the optimal order quantity S also rise accordingly.

(4) As the holding cost in OW c_{ho}, the holding cost in RW c_{hr}, the deteriorated item cost c_d or the scale parameter α_o increases, the optimal present value of total cost TC rises, while the optimal stock period in OW t_o, the optimal replenishment cycle T, the optimal stock period in RW t_r and the optimal order quantity S decrease.

(5) When the shape parameter β_o rises, the optimal present value of total cost TC declines. And the trend for other decision variables is not certain.

6. Conclusions

The paper introduces a two-warehouse partial backlogging inventory model, as presented by Yang (2006), to incorporate a ramp-type demand for deteriorating items. The primary objective is to establish the optimal replenishment policy that minimizes the present value of the total relevant cost per unit time. Based on the values of growth period, the stock period in RW, the location parameter of Weibull distribution in OW, and the location parameter of Weibull distribution in RW, there are seven models be discussed. Three Theorems and an algorithm for finding the optimal solution are established. To validate the results of the proposed inventory model, numerical examples and sensitivity analysis are provided.

The numerical findings offer valuable insights for inventory managers as follows: 1) To reduce the optimal present value of total cost, it is essential to decrease the replenishment cost, the holding cost in OW, or the deteriorated item cost, but to increase the shape parameter of Weibull distribution in OW. 2) If there is an increase in the growth period, the slope of demand rate, the capacity of the owned warehouse, or the replenishment cost but a decrease in the holding cost in OW, the holding cost in RW, the deteriorated item cost or the scale parameter of Weibull distribution in OW, it is advisable to increase the optimal order quantity. 3) When the capacity of the owned warehouse or the replenishment cost experiences an increase, the replenishment cycle should be extended; however, if the growth period, the slope of demand rate, the holding cost in OW, the holding cost in RW, the deteriorated item cost or the scale parameter of Weibull distribution in OW increases, the replenishment cycle must be shortened.

To advance research in this area, this model can be extended in various manners. For example, the model can be extended by incorporating a demand rate function that is a function of the sale price or the retailer strategy of offering discounted prices to attract more customers when advance-cash-credit payment is adopted. Furthermore, we could also add advertisement strategy into consideration.

Appendix A. Algorithm for finding the optimal solution

Step 0. Input parameter values.

Step 1. Compare the values of μ, γ_r, and γ_o.

Step 1.1. If $\mu \le r_r \le r_o$, then from Table 3, we know that only TC_{11} and TC_{12} need to be considered. Go to Step 2.

Step 1.2. If $r_r \le \mu \le r_o$, then from Table 3, we know that TC_{13}, TC_{14} and TC_{22} could be considered. Go to Step 4.

Step 1.3. If $r_r \le r_o \le \mu$, then from Table 3, we know that TC_{13}, TC_{21} and TC_{23} could be considered. Go to Step 6.

Step 2. Calculate t_r, t_o, Δ_2 for TC_{11} and TC_{12}.

Step 2.1. By (57a), (15) and (60a) to find t_r, t_o and Δ_2. And then from (30a) to calculate TC_{11}.

Step 2.2. By (57b), (21) and (60b) to find t_r, t_o and Δ_2. And then from (30b) to calculate TC_{12}.

Step 3. Set $TC(T^*) = min\ \{TC_{11}, TC_{12}\}$ and then $T^* = t_o + \Delta_2$ are the optimal solution, and stop.

Step 4. Calculate t_r, t_o, Δ_2 for TC_{13}, TC_{14} and TC_{22}.

Step 4.1. By (61a), (15) and (62a) to find t_r, t_o and Δ_2. And then from (30c) to calculate TC_{13}.

Step 4.2. By (61b), (21) and (62b) to find t_r, t_o and Δ_2. And then from (30d) to calculate TC_{14}.

Step 4.3. By (63b), (46) and (65b) to find t_r, t_o and Δ_2. And then from (56b) to calculate TC_{22}.

Step 5. Set $TC(T^*) = min\ \{TC_{13}, TC_{14}, TC_{22}\}$ and then $T^* = t_o + \Delta_2$ are the optimal solution, and stop.

Step 6. Calculate t_r, t_o, Δ_2 for TC_{13}, TC_{21} and TC_{23}.

Step 6.1. By (61a), (15) and (62a) to find t_r, t_o and Δ_2. And then from (30c) to calculate TC_{13}.

Step 6.2. By (63a), (39) and (65a) to find t_r, t_o and Δ_2. And then from (56a) to calculate TC_{21}.

Step 6.3. By (63c), (52) and (65c) to find t_r, t_o and Δ_2. And then from (56c) to calculate TC_{23}.

Step 7. Set $TC(T^*) = min\ \{TC_{13}, TC_{21}, TC_{23}\}$ and then $T^* = t_o + \Delta_2$ are the optimal solution, and stop.

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Hui-Ling Yang
Department of Intelligent Technology and Application, Hung Kuang University, Taiwan
E-mail: hui@sunrise.hk.edu.tw
Major area(s): Mathematics, Statistics, Inventory control

Chun-Tao Chang
Department of Statistics, Tamkang University, Taiwan
E-mail: chuntao@mail.tku.edu.tw
Major area(s): Inventory Management, Operations Research

Yao-Ting Tseng (Corresponding Author)
Department of Statistics, Tamkang University, Taiwan
E-mail: yaoting.tseng@mail.tku.edu.tw
Major area(s): Biostatistics, Data Science, Statistical Inference

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