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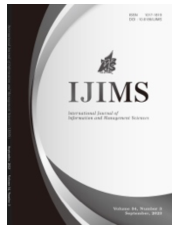

Revisiting Inventory Models with Shortage Demand of Normal Distribution

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Abstract
A recent publication proposed a revised solution for the inventory models presented by the first paper to treat the ordering point as a new variable and a normal distribution for the expected demand during shortage periods. Notably, the recent paper extended the domain of study from almost half to nearly total, marking a significant enhancement. In this paper, we challenge the necessity of their reduced nearly total domain and demonstrate that a pair of lower and upper bounds of their nearly total domain are unnecessary. We rigorously establish the existence and uniqueness of the optimal solution within the original domain. We provide a detailed explanation to reveal the distinct character of this kind of solution approach. Our findings show that our refined approach yields conclusive results, potentially attracting practitioners to delve into inventory models with a normal distribution for the expected demand during shortage periods.

1. Introduction

Businesses and corporationsmust make properdecisions to secure their survival and prosper in today's competitive world. How many items and when to purchase is the central issue for inventory management. Many inventory models under various conditions with respect to different vital factors are developed to help decision-makers decide their best strategies. For three decades, researchers accepted that Moon and Choi (1998) and Hariga and Ben-Daya (1999) are independently constructed inventory models to consider the reordered point as a new decision variable, which can be supported by citations of 187 and 285 times, respectively. On the other hand, only eight papers have been referred to Horowitz and Daganzo (1986) to indicate that academic society almost forgot Horowitz and Daganzo (1986) which is thefirst paper to consider the reordered point as a new decision variable. After examinating Horowitz and Daganzo (1986), we found that they could only handle some domains for their constructed inventory model. Then, they applied a graphical method to locate the optimal solution for 48% of regions of their original domain. Recently, Chuang et al. (2018) tried to arouse the attention

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of practitioners by examining Horowitz and Daganzo (1986) to extend the search domain,which was improved from 48% to 91%. The main goal of this paper is to provide a further revision of Horowitz and Daganzo (1986) and Chuang et al. (2018) such that academic society will pay attention to Horowitz and Daganzo (1986) as the first paper for a hot research topic, to treat the reorder point as a decision variable, in inventory systems. There are eight papers have cited Horowitz and Daganzo (1986) in their references. Except Chuang et al. (2018), the other seven papers only mentioned Horowitz and Daganzo (1986) in their introduction, without any improvement. Interested readers please refer to Chuang et al. (2018) for the other papers.

The purpose of this paper is twofold. First, we show that under the whole searching domain proposed by Horowitz and Daganzo (1986), the optimal solution exists and is unique. Chuang et al. (2018) mentioned that they covered 91% of the search domain to improve the coverage ratio of Horowitz and Daganzo (1986) with 48%. Our improvement is a 100 % coverage ratio for the searching domain. Second, we clearly explain Chuang et al. (2018) to indicate that their solution is correct but incomplete. Consequently, the shrinking domain with two sophisticated boundaries constructed by Chuang et al. (2018) becomes unnecessary. The remainder of this paper is organized as follows.

In Section 2, we compare our results with those of previous studies. In Section 3, we provide notation and assumptions. We review the inventory model of Horowitz and Daganzo (1986) and Chuang et al. (2018) in Section 4. We briefly explain the derivation of a pair of lower and upper bounds proposed by Chuang et al. (2018). We point out that the assertion of Chuang et al. (2018) that the optimal solution is far away from the pair of bounds is not supported by their Example 2. Section 5 provides our improvement in determining the optimal solution for the entire domain. Two numerical Examples in Horowitz and Daganzo (1986) and Chuang et al. (2018) support our claim off $\left(\sqrt{3}\sigma_L/2\right) > 0$ and $g(\Omega) > 0$. Section 6 explains the difficulty in front of the traditional solution method to motivate the solution approach proposed by Chuang et al. (2018) and this paper. We conclude our findings in Section 7.

2. The Problem Descriptions

In the following, we mention the main contribution of our results.

(i) Horowitz and Daganzo' solution approach (1986) only covered 45% of the domain areas proposed by Horowitz and Daganzo (1986).

(ii) Chuang et al. (2018) developed an improved solution method to cover 91% of domain areas.

(iii) In this paper, we showed that the optimal solution for the entire domain exists and is unique.

Horowitz and Daganzo (1986) worked on mathematical methods to derive the search domain for the optimal solution for their proposed inventory model. However, they needed help to finish their analytic examination, and then they applied a graphical method to find the optimal solution. Chuang et al. (2018) and this paper concentrated on analytic procedures to locate the optimal solution. However, Chuang et al. (2018) tried to find the partial domain where the first derivative of the objective function is strictly increasing to guarantee the optimal solution'sexistence and uniqueness. Consequently, Chuang et al. (2018) only considered 91% of the entire domain. We examined the entire domain to ensure that the optimal solution exists and is unique.

The research gap is that up to now, there has yet to be theoretical support for the inventory model developed by Horowitz and Daganzo (1986) to prove that the minimum solution exists and is unique.

Moon and Choi (1998) is the second paper to treat the reordered point as a new decision variable, and then Hariga and Ben-Daya (1999) is the third paper. There are 187 papers to cite Moon and Choi (1998) in their References. On the other hand, there are 285 papers referred to Hariga and Ben-Daya (1999). On the contrary, only eight articles mention Horowitz and Daganzo (1986). Moreover, Chuang et al. (2018) is the only paper that tried to provide a mathematical foundation for the inventory model developed by Horowitz and Daganzo (1986). This paper will provide a mathematical foundation for the existence and uniqueness of the minimum solution of the inventory model proposed by Horowitz and Daganzo (1986) to fulfill the research gap.

Only one paper, Luo et al. (2020), cited Chuang et al. (2018) in its Reference. However, Luo et al. (2020) worked on Wu and Ouyang (2001) and Tung et al. (2010) to revise their iterative algorithms. Chuang et al. (2018) appeared in a long list of papers during the literature review by Luo et al. (2020). Hence, we claim that the incomplete theoretical foundation proposed by Chuang et al. (2018) did not arouse attention to the original article of Horowitz and Daganzo (1986).

3. Notation and Assumptions

To compare with Horowitz and Daganzo (1986) and Chuang et al. (2018), we use the same notation and assumptions as theirs.

D is the annual demand (parts/year).

$\overline{E}_L$ is the forecasted demand during the lead time.

F is the fixed freight cost per regular shipment.

G is the fixed freight cost per shipment expedited because of shortage.

σ_L is the standard derivation of the difference between predicted and actual demand (the measure of uncertainty) during the lead time.

k is the safety factor: the multiple of σ_L that determines the safety stock, a decision variable.

L is the lead time.

P is thepart value.

$p(k)$ is the probability of shortages during the lead time.

Q is the shipment size (number of parts), a decision variable.

$C(k,Q)$ is the total annual cost, the objective function.

$k(Q)$ is the solution of $\partial C(k,Q)/\partial k = 0$, to express k as a function of Q.

$\boldsymbol{R}$ is the inventory cost per dollar per year.

S is the safety stock.

s is the reorder point.

Assumptions of Horowitz and Daganzo (1986), Chuang et al. (2018), and our abbreviation are listed below:

1. There is only one item in the inventory model.

2. The probability of shortages follows a normal distribution, with $p(k)=\Phi(-k)$, where Φ is the cumulative distribution function of a unit normal random variable.

3. All shortages are completely backlogged.

4. To simplify the expression, we assume that $\Omega = DG\big/\sqrt{2\pi}PR\sigma_L$.

5. $g(Q)=\left(PRQ^2/DG\right)-(F/G)-\Phi(-k(Q))$ is an auxiliary function assumed by Chuang et al. (2018).

6. $f(Q)=\ln(\Omega/Q)-\sigma_L^2/8Q^2$ is the second auxiliary function assumed by Chuang et al. (2018).

4. Brief Review of Previous Results

We recall the original problem of an inventory model by Horowitz and Daganzo (1986) with the objective function,

$$C(k,Q)=(DF/Q)+PRQ+kPR\sigma_L+(DG\Phi(-k)/Q), \tag{4.1}$$

for $k \geq 0$ and $Q>0$, where k is the safety and Q is shipment size. For interested readers, please refer to Horowitz and Daganzo (1986) for derivations of their inventory model.

They solved $\partial C(k,Q)/\partial k=0$ to derive that

$$k^*(Q)=\sqrt{2\ln(\Omega/Q)}, \tag{4.2}$$

Where Ω is an auxiliary abbreviation to simplify the expression denoted by us, with

$$\Omega = DG\big/\sqrt{2\pi}PR\sigma_L. \tag{4.3}$$

Based on Equation (4.2), for $k^*(Q)\geq 0$, Horowitz and Daganzo (1986) derived an upper bound for Q as

$$Q \leq DG\big/\left(\sqrt{2\pi}PR\sigma_L\right). \tag{4.4}$$

They plugged $k^*(Q)$ of Equation (4.2) into Equation (4.1) to change the objective function with only one variable, Q, then

$$C(Q)=(DF/Q)+PRQ+PR\sigma_L k^*(Q)+\left(DG\Phi(-k^*(Q))/Q\right). \tag{4.5}$$

Horowitz and Daganzo (1986) derived a very tedious solution procedure that had been discussed by Chuang et al. (2018) to motivate Chuang et al. (2018) to develop a new solution approach. For interested readers, please refer to Chuang et al. (2018) for a detailed discussion of the solution procedure of Horowitz and Daganzo (1986).

Chuang et al. (2018) found that

$$C'(Q) = \left(-DF/Q^2\right) + PR - \left(DG\Phi\left(-k^*(Q)\right)/Q^2\right). \tag{4.6}$$

Motivated by Equation (4.6), Chuang et al. (2018) assumed their first auxiliary function, $g(Q)$, as

$$g(Q) = \left(PRQ^2/DG\right) - (F/G) - \Phi(-k(Q)), \tag{4.7}$$

with

$$C'(Q) = \left(DG/Q^2\right)g(Q). \tag{4.8}$$

The original goal of Chuang et al. (2018) is to prove the existence and uniqueness of the optimal solution, that is, there is a unique root for $g(Q) = 0$, with the domain,

$$0 < Q \le \Omega = DG/\sqrt{2\pi}PR\sigma_L. \tag{4.9}$$

However, owing to technical problems, Chuang et al. (2018) could not handle the entire domain of Equation (4.9), and then Chuang et al. (2018) claimed that $g(Q) = 0$ has a unique solution for a shrunk domain,

$$Q_L < Q < Q_U, \tag{4.10}$$

Where Q_L and Q_U are a pair of lower and upper bound for Q, with a lower bound,

$$Q_L = \sigma_L/2, \tag{4.11}$$

and an upper bound,

$$Q_U = 23\Omega/25. \tag{4.12}$$

In the following, we provide a brief review of how Chuang et al. (2018) find the findings of Equations (4.11) and (4.12).

Chuang et al. (2018) tried to locate a lower bound, Q_U and an upper bound, Q_U with $g(Q_L)\,g(Q_U) < 0$ and $g'(Q) > 0$, for $Q_L < Q < Q_U$. They had discussed that $\lim_{Q\to 0} g(Q) < 0$ to imply the possibility of $g(Q_L) < 0$ such that their goal is revised as

$$g(Q_L) < 0, \tag{4.13}$$

$$g(Q_U) > 0, \tag{4.14}$$

and

$$g'(Q) > 0, \tag{4.15}$$

for $Q_L < Q < Q_U$.

Chuang et al. (2018) derived

$$g'(Q) = (2PRQ/DG) - \left(e^{-\left(k^*(Q)\right)^2/2} \Big/ Q\sqrt{4\pi \ln(\Omega/Q)}\right), \tag{4.16}$$

and then they claimed that to prove $g'(Q) > 0$ for $Q_U > Q > Q_L$ is equivalent to verify

$$\ln(\Omega/Q) > \sigma_L^2/8Q^2, \tag{4.17}$$

for $Q_U > Q > Q_L$, since $e^{(k^*(Q))^2/2} = \Omega/Q$, based on Equation (4.2).

Motivated by Equation (4.17), Chuang et al. (2018) assumed their second auxiliary function as

$$f(Q) = \ln(\Omega/Q) - \sigma_L^2/8Q^2\ , \tag{4.18}$$

such that to show $g'(Q) > 0$ for $Q_U > Q > Q_L$ is converted to derive

$$f(Q) > 0, \tag{4.19}$$

for $Q_U > Q > Q_L$.

Chuang et al. (2018) obtained

$$f'(Q) = [(\sigma_L/2) - Q][(\sigma_L/2) + Q]/Q^3 \tag{4.20}$$

They wanted $f'(Q) < 0$ to imply $f(Q)$ decreasing for $Q_U > Q > Q_L$ such that the inequality of Equation(4.19) can be simplified as $f(Q_U) > 0$.

Referring to Equation (4.20), Chuang et al. (2018) wanted $(\sigma_L/2) - Q < 0$, to derive a possible lower bound as $(\sigma_L/2) \le Q$.

Based on the above discussion, Chuang et al. (2018) changed their goal from Equation (15) to the following new setting, $g(Q_L) < 0$, $g(Q_U) > 0$,and

$$f(Q_U) > 0. \tag{4.21}$$

Based on data from Horowitz and Daganzo (1986), Chuang et al. (2018) claimed that $g(\sigma_L/2) < 0$, yielding that $Q_L = \sigma_L/2$, is an acceptable lower bound.

On the other hand, Chuang et al. (2018) showed that if $Q_U = 23\Omega/25$, then $g(Q_U) > 0$ and $f(Q_U) > 0$, are both satisfied to imply that $Q_U = 23\Omega/25$ is an acceptable upper bound.

Under their shrunk domain, $\sigma_L/2 = Q_L \le Q \le Q_L = 23\Omega/25$, Chuang et al. (2018) proved that the optimal solution of Equation (4.5)is unique.

Chuang et al. (2018) mentioned that their lower bound, with $Q_L = (\sigma_L/2)$ satisfy $Q_L > 0$ and their upper bound, $Q_U = 23\Omega/25$ satisfies $Q_U < \Omega$ such that they covered about 91% of the original searching domain.

Chuang et al. (2018) pointed out that

(i) Horowitz and Daganzo (1986) applied a graphic method to find the optimal solution that only covered 48% of the original search domain.

(ii) The optimal solution of Example 2 is $Q^* = 2570$ that is far away from two boundaries $Q_L = 50$ and $Q_U = 23857$, as proposed by Chuang et al. (2018).

Consequently, they shrank the search domain to a 91% of the entire domain to guarantee the existence and uniqueness of the optimal solution that is an acceptable approach.

5. Our Improvement

Chuang et al. (2018) tried to apply the Intermediate Value Theorem to prove that $C'(Q, k(Q)) = 0$ has a unique solution for $0 < Q \le DG/\left(\sqrt{2\pi}PR\sigma_L\right)$ or for a reasonable sub-domain.

It is the existence and uniqueness problem for the optimal solution. We recall the Intermediate Value Theorem as follows:

If $h(x)$ is a continuous function for $a \le x \le b$ and $h(a)h(b) < 0$, then there is a point , say c, satisfying $h(c) = 0$.

We know that the Intermediate Value Theorem is related to the existence of the root that is not related to the uniqueness of the root.

Chuang et al. (2018) claimed that they shrank the searching domain from $(0, \Omega]$ to $[Q_L, Q_U]$ with three desired properties: (a) $g(Q_L) < 0$, (b) $g(Q_U) > 0$, and (c) $g'(Q) > 0$ for $Q_L < Q < Q_U$, where $g(Q)$ is an auxiliary function satisfying $C'(Q) = (DG/Q^2)g(Q)$ such that $C'(Q) = 0$ and $g(Q) = 0$ have the same solutions. The Intermediate Value Theorem guarantees the existence of a solution for $g(Q) = 0$ that is not related to the uniqueness of the solution, such that the claim of Chuang et al. (2018) contained a questionable result.

In the following, we provide a revision. Based on the property (c), $g(Q)$ is strictly increasing for $Q_L < Q < Q_U$ to ensure the uniqueness of the solution for $g(Q) = 0$.

We check their above assertion of Example 2 to compare the optimal solution, $Q^* = 2570$ with two boundaries proposed by Chuang et al. (2018), $Q_L = 50$ and $Q_U = 23857$ to find that $2570/50 = 51.4$ and $23857/2570 = 9.3$ both are far away from one to indicate that their assertion, "Q^* is far away from their boundaries Q_L and Q_U" is supported by their Example 2.

However, we check Example 1 of Chuang et al. (2018) with the optimal solution, $Q^* = 2663$, and two boundaries, $Q_L = 200$, and $Q_U = 5964$ to derive that $2663/200 = 13.3$, and $5964/2663 = 2.2$. Owing to 2.2 being close to one, it reveals that their assertion "Q^* is far away from their boundaries Q_L and Q_U" is not supported by their Example 1.

Motivated by the above discussion, the purpose of our improvement is twofold. First, we will show that under the original entire domain, $0 < Q \le \Omega$, then $C'(Q) = 0$ has a unique solution, under data of two numerical examples in Horowitz and Daganzo (1986) and Chuang et al. (2018). Second, we show that the finding of Chuang et al. (2018) is correct but incomplete, such that our resultsprovide a complete solution.

We refer to the findings of Equations (4.16) and (4.18), and then we rewrite $g'(Q)$ as

$$g'(Q) = \left(2\sqrt{2}PRQ/DG(\ln(\Omega/Q))^{1/2}\right)\left[\sqrt{2\ln(\Omega/Q)} + (\sigma_L/2Q)\right]f(Q) \tag{5.1}$$

such that $g'(Q)$ and $f(Q)$ have the same sign.

From Equation (4.20), we obtain $f'(Q) > 0$ for $0 < Q < \sigma_L/2$ and $f'(Q) < 0$ for $\sigma_L/2 < Q < \Omega$ to imply that $f(Q)$ increases for $0 < Q \le \sigma_L/2$ and then decreases for $\sigma_L/2 \le Q \le \Omega$.

From Equations (4.3) and (4.18), we rewrite Equation (4.18) as follows,

$$f(Q) = \ln(DG) - \ln\left(\sqrt{2\pi}PR\sigma_L\right) - \ln Q - \left(\sigma_L^2/8Q^2\right). \tag{5.2}$$

Now, we compute $\lim_{Q\to 0} -\ln Q - \left(\sigma_L^2/8Q^2\right)$ to imply

$$-\ln Q - \left(\sigma_L^2/8Q^2\right) = \left(8Q^2\ln Q + \sigma_L^2\right)/\left(-8Q^2\right). \tag{5.3}$$

By the Hospital rule, we evaluate

$$\lim_{Q\to 0} Q^2 \ln Q = \lim_{Q\to 0} \ln Q \big/ Q^{-2} = \lim_{Q\to 0} Q^{-1} / \left(-2Q^{-3}\right) = 0, \tag{5.4}$$

to yield

$$\lim_{Q\to 0} f(Q) = -\infty. \tag{5.5}$$

From Equation (4.20), we know that $f(Q)$ decreases for $\sigma_L/2 \le Q \le \Omega$, and $f(Q_U) > 0$ of Equation (5.2) derived by Chuang et al. (2018), then we derive

$$f(\sigma_L/2) > 0. \tag{5.6}$$

On the other hand, we find

$$f(\Omega) = -\sigma_L^2 / 8\Omega^2 < 0. \tag{5.7}$$

We apply our results of $\lim_{Q\to 0} f(Q) = -\infty$, $f(Q)$ increases for $0 < Q \le \sigma_L/2$, and $f(\sigma_L/2) > 0$ to imply that there is a unique point, say Q_1 with $f(Q_1) = 0$ and $f(Q) < 0$ for $0 < Q < Q_1$, and $f(Q) > 0$, for $Q_1 < Q < \sigma_L/2$.

Similarly, we use our findings of $f(\sigma_L/2) > 0$, $f(\Omega) < 0$ and $f(Q)$ decreases for $\sigma_L/2 \le Q \le \Omega$, to yield that there is a unique point, say Q_2 with $f(Q_2) = 0$ and $f(Q) > 0$, for $\sigma_L/2 < Q < Q_2$, and $f(Q) < 0$, for $Q_2 < Q < \Omega$.

For ordinary readers, we begin to sketch the graph of $g(Q)$.

Owing to Equation (5.1), we know that $g'(Q)$ and $f(Q)$ have the same sign. Hence, we will first sketch the graph of $f(Q)$.

We recall Equation (5.1), then we derive that

$$f''(Q) = \left[Q - \left(\sqrt{3}\sigma_L/2\right)\right]\left[\left(\sqrt{3}\sigma_L/2\right) + Q\right] \big/ Q^4. \tag{5.8}$$

Based on Equation (5.8), we find that $f(Q)$ is concave down for $0 < Q < \sqrt{3}\,\sigma_L/2$, and is concave up for $\sqrt{3}\sigma_L/2 < Q < \Omega$.

We compute $f\left(\sqrt{3}\sigma_L/2\right)$ to imply that

$$f\left(\sqrt{3}\sigma_L/2\right) = \ln\left(\frac{\sqrt{2}DG}{\sqrt{3\pi}PR\sigma_2^2}\right) - \frac{1}{6}. \tag{5.9}$$

We combine our findings in the following theorem.

Theorem 1. There are two points Q_1 and Q_2 with $0 < Q_1 < \sigma_L/2 < Q_2 < \Omega$ and $f(Q_1) = 0 = f(Q_2)$ such that $f(Q) < 0$, for $0 < Q < Q_1$, and $Q_2 < Q < \Omega$. Moreover, $f(Q) > 0$, for $Q_1 < Q < Q_2$.

We recall that Chuang et al. (2018) derived that $f(23\Omega/25) > 0$ to imply

$$23\Omega/25 < Q_2 < \Omega. \tag{5.10}$$

We obtain that based on Equation (5.1), $g'(Q)$ and $f(Q)$ have the same sign. Hence, applying Theorem 1, we derive that $g(Q)$ decreases for $0 < Q < Q_1$, and $Q_2 < Q < \Omega$. On the other hand, $g(Q)$ increases for $Q_1 < Q < Q_2$.

Chuang et al. (2018) already showed that $\lim_{Q \to 0} g(Q) = -F/G < 0$ such that for $0 < Q < Q_1$, $g(Q)$ further decreases to its minimum $g(Q_1)$.

Chuang et al. (2018) already showed that $g(23\Omega/25) > 0$ to imply

$$g(Q_2) > 0, \tag{5.11}$$

since we know Equation (5.10) and $g(Q)$ increases, for $Q_1 < Q < Q_2$.

Wefind that $g(Q)$ increases, for $Q_1 < Q < Q_2$. We recall that $g(Q_1) < 0$, increasing to $g(\sigma_L/2) < 0$ (derived by Chuang et al. (2018)) and then further increases until $g(Q_2) > 0$ such that there is a unique point, say Q^Δ satisfying

$$g(Q^\Delta) = 0, \tag{5.12}$$

where $Q_1 < \sigma_L/2 < Q^\Delta < Q_2$.

From our Theorem 1, we know that $g(Q)$ decreases for $Q_2 < Q < \Omega$ to its minimum, $g(\Omega)$.

In the following, based on numerical examples of Horowitz and Daganzo (1986), and Chuang et al. (2018), we will show that $g(\Omega) > 0$. We derive

$$g(\Omega) = (DG/2\pi PR\sigma_L^2) - (F/G) - (1/2). \tag{5.13}$$

We recall data from Examples 1 and 2 of Horowitz and Daganzo (1986) and Chuang et al. (2018), with $D = 52000$ parts per year, $F = \$2500$ per shipment, $G = \$2500$ per shipment, $P = \$100$ per part, $R = 0.2$ per year (20%), $L = 7$ days, $\sigma_L = 400$ parts for the first example, and $\sigma_L = 100$ parts for the second example to derive that with $\sigma_L = 400$,

$$g(\Omega) = 4.966 > 0, \tag{5.14}$$

and with $\sigma_L = 100$,

$$g(\Omega) = 101.951 > 0. \tag{5.15}$$

Based on Equations (5.14) and (5.15), based on two numerical examples of Horowitz and Daganzo (1986) and Chuang et al. (2018), we derive that

$$g(\Omega) > 0. \tag{5.16}$$

Moreover, based on the first example of Horowitz and Daganzo (1986), and Chuang et al. (2018) with $\sigma_L = 400$, we derive that

$$f(\sqrt{3}\sigma_L/2) = \ln\left(\frac{\sqrt{2}DG}{\sqrt{3\pi}PR\sigma_2^2}\right) - \frac{1}{6} = 2.763. \tag{5.17}$$

On the other hand, according to the first example of Horowitz and Daganzo (1986) and Chuang et al. (2018) with $\sigma_L = 100$, we find that

$$f\left(\sqrt{3}\sigma_L/2\right) = \ln\left(\frac{\sqrt{2}DG}{\sqrt{3\pi}PR\sigma_2^2}\right) - \frac{1}{6} = 5.535. \quad (5.18)$$

We observe the findings of Equations (5.17) and (5.18), based on the examples of Horowitz and Daganzo (1986) and Chuang et al. (2018), we claim that

$$f\left(\sqrt{3}\sigma_L/2\right) > 0. \quad (5.19)$$

We combine our derivations in the followingtheorem.

Theorem2. Based on data from two numerical examples in Horowitz and Daganzo (1986) and Chuang et al. (2018), there is a unique point, say Q^Δ, with $g(Q^\Delta) = 0$ such that $g(Q) < 0$, for $0 < Q < Q^\Delta$ and $g(Q) > 0$, for $Q^\Delta < Q \le \Omega$.

Based on our discussion above, we recall the results after Equations (5.7) and (5.8), then sketch the graph of $f(Q)$ and $g(Q)$ in the following.

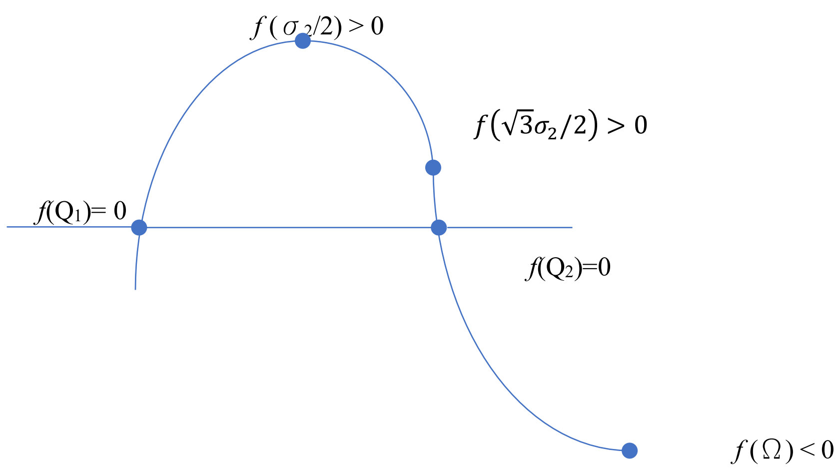

Fig 1. *The Graph of* $f(Q)$.

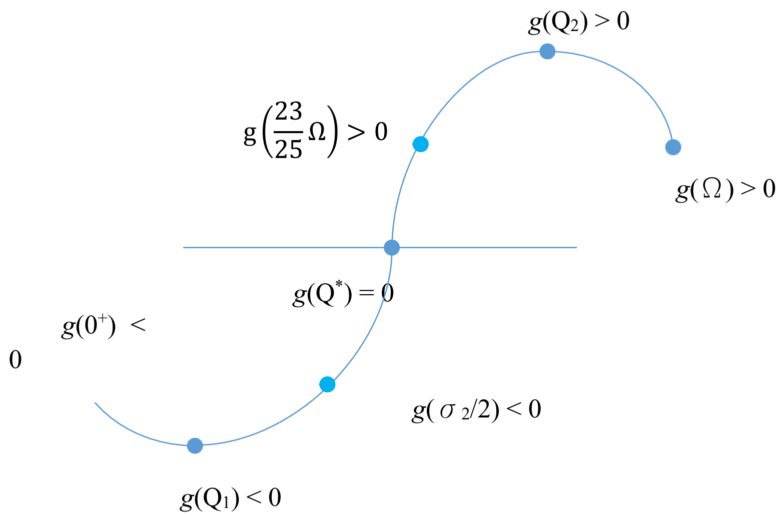

Fig 2. *The Graph of* $g(Q)$.

We recall Equation (8) to imply that $C'(Q)$ and $g(Q)$ have the same sign. Hence, according to our Theorem 2, we obtain the following theorem.

Theorem 3. Q^{Δ} is the minimum solution for $C(Q)$.

Based on Theorem 3, we claim that the solution of $g(Q)=0$ is the optimal solution, which will be denoted as Q^*.

Therefore, we show that the optimal solution exists and is uniqueunder the original setting of $0<Q\leq\Omega$.

Chuang et al. (2018) derived the optimal solution for $[\sigma_L/2, 23\Omega/25]$, which is 91% of the entire domain. This study proves that the optimal solution proposed by Chuang et al. (2018) is the genuine optimal solution for the entire domain. Our results show that researchers will not worry about those optimal solutions occurring in $(0,\sigma_L/2)$ or $(23\Omega/25,\infty)$.

6. The Distinct Character of This Solution Approach

Chuang et al. (2018) pointed out that Horowitz and Daganzo (1986) used a graphical method to locate the minimum solution for the sub-domain, $0<Q<\frac{5\sqrt{2\pi}}{26}\Omega$ which is about 48% of the original domain, $0<Q\leq\Omega$. Based on $\partial C(k,Q)/\partial k=0$, Horowitz and Daganzo (1986) already derived the relation of Equation (4.2). Chuang et al. (2018) plugged the relation of Equation (4.2) to convert a two-variable objective function, $C(k,Q)$, into a one-variable objective function, $C(k(Q),Q)$, denoted as $C(Q)$. The optimal solution satisfies $C'(Q)=0$. Directly sketching the graph of $C'(Q)$ is a difficult task. Chuang et al. (2018) constructed an auxiliary function, $g(Q)$, with the same sign as $C'(Q)$. Directly sketching the graph of $g(Q)$ is still a difficult task. Chuang et al. (2018) constructed a second auxiliary function, $f(Q)$, which has the

same sign as $g'(Q)$. Chuang et al. (2018) focused on finding a sub-domain; within it, $g(Q)$ is strictly increasing, and then Chuang et al. (2018) developed a sub-domain as $[\sigma_L/2, 23\Omega/25]$.

By our analysis, the increasing sub-domain of $g(Q)$ should be $[Q_1, Q_2]$, with $Q_1 < \sigma_L/2$, and $23\Omega/25 < Q_2$. Consequently, the lower bound, $\sigma_L/2$, and the upper bound, $23\Omega/25$ proposed by Chuang et al. (2018), become incomplete and redundant.

Moreover, for the other two sub-domains, $0 < Q < Q_1$ and $Q_2 < Q < \Omega$, $g(Q)$ is decreasing. However, $g(Q)$ decreases in these two sub-domains that will not influence the uniqueness of solution for $g(Q) = 0$ in the entire domain, $0 < Q \le \Omega$. Consequently, we can claim that Chuang et al. (2018) tried to find a sub-domain in which $g(Q)$ is an increasing function that is a questionable solution direction.

On the other hand, we admit the contribution of Chuang et al. (2018) to construct two auxiliary functions, $g(Q)$ and $f(Q)$, to cleverly avoid the difficult task of examining the graph of $C'(Q)$ and analyzing $g''(Q)$, and $g''(Q)=0$.

In several inventory models, authors claimed that their objective functions have the convex property such that their minimum solution exists. The convex property will imply that the second derivative is positive, and then the first derivative is an increasing function. Referring to this inventory model proposed by Horowitz and Daganzo (1986), $g(Q)$ is an increasing function. Based on our Theorem 1, we know the zeros of $f(Q)$ and then we sketch the graph of $f(Q)$ at Figure 1. According to our Theorem 2, we know the zeros of $g(Q)$ and then we sketch the graph of $g(Q)$ at Figure 2. Owing to $C'(Q)$ and $g(Q)$ having the same sign, we derive a unique solution, denoted as Q^{Δ}, the optimal solution of our objective function, $C(Q)$. In this paper, we demonstrate that the inventory model proposed by Horowitz and Daganzo(1986) did not have the convex property because the convex property will imply that $g(Q)$ is an increasing function. Referring to Figure 2, we derive that $g(Q)$ decreases for $0<Q<Q_1$ and $Q_2 < Q \le \Omega$. Moreover, we can still prove that the critical point exists and is unique, which is the minimum point.

The benefit of the analytic method proposed by Chuang et al. (2018) and this paper to the graphical method proposed by Horowitz and Daganzo (1986) is illustrated by Chuang et al. (2018). Based on two numerical examples mentioned by Horowitz and Daganzo (1986), the analytic method produces fewer cars against global warming and also saves money.

7. Conclusion

In this comprehensive investigation, we have thoroughly explored the entire search domain, $(0,\Omega]$, enabling us to unequivocally confirm the presence and singularity of the optimal solution. Consequently, the previously proposed lower and upper bounds, $Q_L = \sigma_L/2$ and $Q_U = 23\Omega/25$ as articulated by Chuang et al. (2018), become redundant. In conclusion, we extend an intriguing challenge for future research: the direct verification of two outcomes, $f(\sqrt{3}\sigma_L/2)>0$, and $g(\Omega)>0$, supported by numerical examinations from a managerial perspective, presents a captivating avenue for exploration. In this paper, we only applied the numerical method to support our assertions of $f(\sqrt{3}\sigma_L/2) > 0$, and $g(\Omega)>0$. To develop a complete solution structure, in the future, to consider $f(\sqrt{3}\sigma_L/2) \le 0$, or $g(\Omega)\le 0$, will be an exciting research problem.

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