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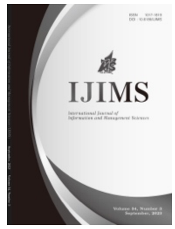
International Journal of
Information
and
Management
Sciences

International Journal of
Information and Management Sciences

35 (2024), 119-138. DOI:10.6186/IJIMS.202406_35(2).0002



Behavior optimization of two-stage vacation fluid queue with two types of parallel customers

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Keywords

Fluid queue, two-stage vacation, parallel customers, Nash equilibrium strategy, average social benefit per unit time.

Abstract

Based on the order processing process of automated guided transport vehicles (AGVs) in the warehouse, this paper constructs a two-stage vacation fluid queuing model with two types of parallel customers, where the arrival processes of the two types of fluids are independent of each other, and the second type of fluid enters with different probabilities depending on the state of the buffer. Based on the conditional expectation formula, the mean sojourn time of the fluid is obtained. The stationary distribution of fluid level is derived by solving the differential equations. Furthermore, the Nash equilibrium strategy of fluid and the average social profits per unit time is acquired. The SOA algorithm is designed to find the social optimal joining threshold and the maximum social profit. Numerical analysis demonstrates the influence of key parameters on social benefit. The results of this paper can broaden the research scope of the economic analysis of fluid queue and provide theoretical reference for optimization management of logistics system.

1. Introduction

Different from the classical queuing model, discrete customers are replaced by continuous fluid in a fluid queuing model. Therefore, fluid queuing model has a wide range of applications in the current era of big data, such as network transmission(Stern & Elwalid, 1991), cloud logistics system (Latouche & Taylor, 2009) and management industry (Ahn & Ramaswami, 2003).

Combining queuing theory and game theory, the economic analysis of queuing model focus on the equilibrium balking strategy and social optimal joining strategy. Customers (fluid) will get certain payoff after being served, and need to spend a certain cost during their sojourn period. The net benefit of a rational customer was regarded the criterion to judge whether to enter the buffer. Each customer only considers his own net benefit, regardless of the profit of others. Customers game each other, the optimal strategy of individual benefit may not be the

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optimal strategy of social profit. Therefore, more and more scholars begin to analyze fluid queue from the perspective of economics.

Rajagopal et al. (1995) considered a stochastic fluid queuing model with infinite capacity, constant output rate and adjustable input rate, derived the social optimal joining threshold and the optimal charging strategy. Maglaras (2006) analyzed the fluid queuing model with dynamic pricing problem, and gave the equilibrium balking strategy and the social optimal entry strategy for different types of customers. Kesselman and Leonardi (2012) studied a precise grouping fluid queuing model and gave the Nash equilibrium strategy for customers. Economou and Manou (2016) discussed the fluid queuing model with two service modes under the fully observable case, and equilibrium balking strategy for customers was investigated. Barron (2018) derived the social optimal threshold of the total storage level for a fluid queue. Kelly and Yudovina (2018) provided various high-frequency trading strategies in a fluid queuing model, and discussed the Nash equilibrium strategy among high-frequency traders when the continuous time market is frequently auctioned. Xu et al. (2021) studied the repairable fluid queuing model with threshold-controllable arrival rate under fully observable case and almost observable case, an exponential utility function of social benefit was constructed and the individual equilibrium balking strategy was obtained. Wang and Xu (2021) studied a fluid queuing model with limited buffer capacity and adjustable service rate from the perspective of economics. Liu et al. (2020) discussed a fluid model with parallel customers and breakdowns, obtained the equilibrium individual balking strategy and the social optimal enter strategy. Wang and Xu (2024) investigated the equilibrium strategy in a fluid model with two types of parallel customers and delayed repair.

The classical vacation policy means that the queuing system does not work during the vacation period, while the working vacation strategy means that the queuing system does not stop service completely and works at a low speed. Zhang and Xu (2022) analyzed a vacation fluid queue with controllable fluid outflow rate from the view of economics. Xu and Wang (2021) conducted an economic analysis of the fluid queuing model with working vacation under the fully observable case and almost observable case. Wang and Xu (2018) introduced set-up time and working vacation policy into a fluid queue, and derived the social optimal threshold of the fluid model.

Order picking is a crucial link in logistics systems, and the timeliness of product delivery has become increasingly important to modern enterprise. Warehousing operations have gradually evolved into order fulfillment centers (OFC), which focus on providing small-batch, multi-batch shipments to individual customers. Automated guided vehicles (AGVs) as a novel mode of goods transportation has greatly enhanced transporting efficiency. In the order picking system, once a customer places an order, the OFC promptly allocates the task to the warehouse. The AGVs within the warehouse transport shelves containing the desired items to the picking stations, where order pickers efficiently select the products based on the order information.

A large number of orders enter the OFC continuously like the fluid, and the sorted goods are continuously sent out. Orders are categorized into two types, corresponding to ordinary goods and fragile goods respectively. Taking into account the risk of fragile items, orders for fragile items placed during work vacation period and vacation period will be reduced by probability.

Based on the above research and considering the working mode of AGVs, this paper constructs a two-stage vacation fluid queuing model with two types of parallel customers, develops solving method for average sojourn time and stationary distribution of fluid level, and derives the individual equilibrium balking strategy, and improves SOA algorithm to obtain optimal social benefit. The study extends the results of the existing literature, enriches the theoretical research framework of fluid models, and expands the application field of fluid model.

The rest of this paper is organized as follows: Section 2 constructs a two-stage vacation fluid queuing model with two kinds of parallel customers. section 3 studies the equilibrium balking strategy of the fluid under the fully observable case. Section 4 derives an explicit expression for the stationary distribution of fluid level, and the average social profits. Section 5 uses numerical analysis to demonstrate the influence of key parameters on social benefits, and SOA algorithm is designed to find the social optimal joining thresholds and the maximum social profit. Section 6 summarizes the content, and puts forward the prospect of the future research work.

2. Model description and assumption

In the two-stage vacation fluid queuing model with two types of parallel customers (fluid), the model assumptions are as follows:

The buffer alternates between three system states, namely working period, working vacation period, and vacation period. The duration of these states follow exponential distributions with parameters θ_0, θ_1 and θ_2, respectively.

Two types of fluid independently flow into the buffer. The first type of fluid represents normal customers and has a constant inflow rate of λ_1. The inflow rate of the second type of fluid is λ_2 ($\neq \lambda_1$), but it flows into the buffer according to different probabilities q_1 and q_2 during the working vacation period and the vacation period respectively.

During the working period, the buffer has an outflow rate of μ_b. After the working period, the buffer switches to the working vacation period, and its outflow rate decreases to μ_v $(\mu_v < \mu_b)$. After the working vacation period, the buffer switches to the vacation period, during which the buffer does not work and there is no outflow. Once the vacation period is over, the buffer re-enters the working period.

Assume that the fluid level in the buffer at time t is denoted by $X(t)$, the state of the buffer at time t is denoted by $I(t)$, and $I(t)=0,1,2$ represents the buffer stays in the working period, working vacation period, and vacation period, respectively. Then the net input rate structure of the buffer can be described as follows

$$\frac{dX(t)}{dt}=\begin{cases}\lambda_1+\lambda_2-\mu_b, & I(t)=0,\, X(t)>0,\\ \lambda_1+q_1\lambda_2-\mu_v, & I(t)=1,\, X(t)>0,\\ \lambda_1+q_2\lambda_2, & I(t)=2,\, X(t)>0,\\ 0, & X(t)=0.\end{cases}$$

On the other hand, based on the alternating renewal process, the steady-state probability distribution of the process $\{I(t), t \geq 0\}$ can be obtained as follows

$$\pi_0 = \frac{\theta_1\theta_2}{\theta_1\theta_2 + \theta_0\theta_2 + \theta_0\theta_1}, \pi_1 = \frac{\theta_0\theta_2}{\theta_1\theta_2 + \theta_0\theta_2 + \theta_0\theta_1}, \pi_2 = \frac{\theta_0\theta_1}{\theta_1\theta_2 + \theta_0\theta_2 + \theta_0\theta_1}.$$

3. Individual equilibrium balking strategy

In the fully observable case, customers have full knowledge of system information, including the state of the buffer and fluid level. Assume that type k fluid gains a reward of R_k after service, and pays the cost of C_k per unit time during the sojourn period. The fluid is willing to enter the buffer only if the net benefit after being served is positive. Assume that the fluid cannot exit before being served. Similarly, they cannot change their decision if they make a decision to refuse to enter the buffer. Because the factors that affect the benefit of fluid in real life are very complex, it is very difficult to discuss them directly when the fluid reaches the buffer, this paper mainly discusses the impact of sojourn time on the benefits of fluid, so we denoted the average benefit per unit time of type k fluid flowing into the buffer when $(X(t), I(t)) = (x, i)$ by

$$\Phi_k(x,i) = R_k - C_k E(S_k(x,i)), k = 1,2; i = 0,1,2. \tag{3.1}$$

where $E(S_k(x,i))$ represents the average sojourn time of type k fluid flowing into the buffer when $(X(t), I(t)) = (x, i)$.

Assume that $R_k > \max_{k=1,2}\{C_k/\mu_v, C_k(1/\theta_2 + 1/\mu_b)\}$ in order to ensure that the fluid is willing to flow into the buffer when the buffer is empty.

Theorem 1. In the case of fully observable, the Nash equilibrium threshold for the two-stage vacation fluid queue with two types parallel customers is (x_{0k}, x_{1k}, x_{2k}). That is , when type k fluid reaches the system at time t and finds the system information (x, i), the fluid will choose to flow into the buffer if $x < x_{ik}$. Otherwise it will balk. These thresholds $x_{ik},\ i = 0,1,2$ respectively are the unique solutions of the following equations

$$\frac{m_0}{m_1 m_2} e^{m_2 x_{0k}} + \left(\frac{1}{\mu_b} - \frac{m_0}{m_1}\right) x_{0k} - \frac{m_0}{m_1 m_2} - \frac{R_k}{C_k} = 0, \quad k = 1,2,$$

$$-\frac{\theta_1\mu_b m_0}{\theta_0\mu_v m_1 m_2} e^{m_2 x_{1k}} + \left(\frac{1}{\mu_b} - \frac{m_0}{m_1}\right) x_{1k} - \frac{\theta_1\mu_b m_0}{\theta_0\mu_v m_1 m_2} - \frac{R_k}{C_k} = 0, \quad k = 1,2,$$

$$\frac{m_0}{m_1 m_2} e^{m_2 x_{2k}} + \left(\frac{1}{\mu_b} - \frac{m_0}{m_1}\right) x_{2k} - \frac{m_0}{m_1 m_2} + \frac{1}{\theta_2} - \frac{R_k}{C_k} = 0, \quad k = 1,2,$$

where $m_0 = 1/\mu_b - (1 + \theta_1/\theta_2)/\mu_v$, $m_1 = 1 + \theta_1\mu_b/\theta_0\mu_v$, $m_2 = -\theta_0/\mu_b - \theta_1/\mu_v$.

Proof. Define that T_i is the remaining time of the buffer in state $i\,(i = 0,1)$ after the fluid enters, then it follows an exponential distribution with parameter θ_i based on the memorylessness of exponential distribution. According to the conditional expectation formula, we obtain

$$E\left(S_k\left(x,0\right)\right)=\int_0^{\frac{x}{\mu_b}}\left(t+E\left(S_k\left(x-\mu_b t,0\right)\right)\right)\theta_0 e^{-\theta_0 t}dt+\frac{x}{\mu_b}e^{\frac{-\theta_0 x}{\mu_b}}, \tag{3.2}$$

$$E\left(S_k\left(x,1\right)\right)=\int_0^{\frac{x}{\mu_v}}\left(t+E\left(S_k\left(x-\mu_v t,2\right)\right)\right)\theta_1 e^{-\theta_1 t}dt+\frac{x}{\mu_v}e^{\frac{-\theta_1 x}{\mu_v}}. \tag{3.3}$$

Taking $v_0=x-\mu_b t$ in (3.2) and $v_1=x-\mu_v t$ in (3.3), we get

$$E\left(S_k\left(x,0\right)\right)=\frac{1}{\theta_0}-\frac{1}{\theta_0}e^{\frac{-\theta_0 x}{\mu_b}}+\frac{\theta_0}{\mu_b}e^{\frac{-\theta_0 x}{\mu_b}}\int_0^x E\left(S_k\left(v_0,1\right)\right)e^{\frac{\theta_0 v_0}{\mu_b}}dv_0 \tag{3.4}$$

$$E\left(S_k\left(x,1\right)\right)=\frac{1}{\theta_1}-\frac{1}{\theta_1}e^{\frac{-\theta_1 x}{\mu_v}}+\frac{\theta_1}{\mu_v}e^{\frac{-\theta_1 x}{\mu_v}}\int_0^x E\left(S_k\left(v_1,2\right)\right)e^{\frac{\theta_1 v_1}{\mu_v}}dv_1 \tag{3.5}$$

Multiplying both sides of (3.4) and (3.5) by $\mu_b e^{\frac{\theta_0 x}{\mu_b}}$ and $\mu_v e^{\frac{\theta_1 x}{\mu_v}}$, respectively, then differentiating with respect to x, we have

$$\frac{dE\left(S_k\left(x,0\right)\right)}{dx}=\frac{-\theta_0}{\mu_b}E\left(S_k\left(x,0\right)\right)+\frac{\theta_0}{\mu_b}E\left(S_k\left(x,1\right)\right)+\frac{1}{\mu_b}, \tag{3.6}$$

$$\frac{dE\left(S_k\left(x,1\right)\right)}{dx}=\frac{-\theta_1}{\mu_v}E\left(S_k\left(x,1\right)\right)+\frac{\theta_1}{\mu_v}E\left(S_k\left(x,2\right)\right)+\frac{1}{\mu_v}. \tag{3.7}$$

Based on the boundary conditions $E\left(S_k\left(0,i\right)\right)=0, i=0,1$, manipulating (3.6) and (3.7) ,we obtain

$$E\left(S_k\left(x,0\right)\right)=\frac{m_0}{m_1 m_2}e^{m_2 x}+\left(\frac{1}{\mu_b}-\frac{m_0}{m_1}\right)x-\frac{m_0}{m_1 m_2},$$

$$E\left(S_k\left(x,1\right)\right)=-\frac{\theta_1\mu_b m_0}{\theta_0\mu_v m_1 m_2}e^{m_2 x}+\left(\frac{1}{\mu_b}-\frac{m_0}{m_1}\right)x-\frac{\theta_1\mu_b m_0}{\theta_0\mu_v m_1 m_2}.$$

From model assumption, we get

$$E\left(S_k\left(x,2\right)\right)=E\left(S_k\left(x,0\right)\right)+\frac{1}{\theta_2}. \tag{3.8}$$

Substituting (3.4), (3.5) and (3.8) into (3.1), we obtain the Nash equilibrium thresholds for the fluid queuing model, then theorem 1 is proved.

4. Social optimal strategy

Customers often focus on maximizing their own benefit and don't concern the profits of group society. The profits of the social group are not necessarily optimal when the benefits of each customer are optimal. If considering a service system from a social perspective, the objective becomes maximizing social welfare, which includes the benefit of all the customers.

4.1 The steady-state probability distribution of fluid levels

Assuming all the type k fluid follows the social optimal strategy $(x_k(0), x_k(1), x_k(2)), k=1,2,$ with model assumption conditions it is generally observed that $x_k(1) < x_k(0)$, $x_k(2) < x_k(0), k=1,2,$ because of the service rate on state 0 is higher than that on status 1 and 2. May as well assume $x_1(i) < x_2(i), i=1,2$. Therefore, there are four possible cases for the thresholds $x_k(1)$ and $x_k(2), k=1,2.$

a) $x_1(1) < x_2(1) < x_1(2) < x_2(2).$ *b*) $x_1(1) < x_1(2) < x_2(1) < x_2(2).$

c) $x_1(2) < x_1(1) < x_2(2) < x_2(1).$ *d*) $x_1(2) < x_2(2) < x_1(1) < x_2(1).$

In the following, the stationary probability distribution of the fluid level in the buffer is discussed in case a). The other cases are not discussed in detail.

Defining the joint probability distribution function of the fluid level in the buffer when it is in state i at time t as

$$F_i(t,x) = P(X(t) \le x, I(t) = i), \quad i = 0,1,2, \quad x \ge 0.$$

thus the steady-state distribution of the fluid level in the buffer at state i is

$$F_i(x) = \lim_{t\to+\infty} P(X(t) \le x, I(t) = i), \quad i = 0,1,2, \quad x \ge 0.$$

Define r_i as the net input rate of the fluid in state i ,then we have

$$r_0 = \frac{x_1(0)}{x_2(0)}(\lambda_1+\lambda_2) + \frac{x_2(0)-x_1(0)}{x_2(0)}\lambda_2 - \mu_b = \frac{x_1(0)}{x_2(0)}\lambda_1 + \lambda_2 - \mu_b,$$

$$r_1 = \frac{x_1(1)}{x_2(1)}(\lambda_1+q_1\lambda_2) + \frac{x_2(1)-x_1(1)}{x_2(1)}q_1\lambda_2 - \mu_v = \frac{x_1(1)}{x_2(1)}\lambda_1 + q_1\lambda_2 - \mu_v,$$

$$r_2 = \frac{x_1(2)}{x_2(2)}(\lambda_1+q_2\lambda_2) + \frac{x_2(2)-x_1(2)}{x_2(2)}q_2\lambda_2 = \frac{x_1(2)}{x_2(2)}\lambda_1 + q_2\lambda_2,$$

and denote $\alpha = x_1(0)/x_2(0)$, $\beta = x_1(1)/x_2(1)$, $\gamma = x_1(2)/x_2(2)$.

If all the fluid follows socially optimal strategies $(x_k(0), x_k(1), x_k(2)), k=1,2$, under condition $x_1(1) < x_2(1) < x_1(2) < x_2(2) < x_1(0) < x_2(0)$, then the steady-state distribution of the fluid level can be discussed in four cases: (1) $r_0 > 0, r_1 > 0;$ (2) $r_0 > 0, r_1 < 0;$ (3) $r_0 < 0, r_1 > 0;$ (4) $r_0 < 0, r_1 < 0.$

Theorem 2. when $r_0 > 0, r_1 > 0$, then

$$F_0(x) = \begin{cases} 0, & x < x_2(1), \\ k_{10} + k_{11}e^{h_{11}x} + k_{12}e^{h_{12}x}, & x_2(1) < x \le x_2(0), \\ \pi_0, & x > x_2(0), \end{cases}$$

$$F_1(x) = \begin{cases} 0, & x < x_2(1), \\ k_{10}\dfrac{\theta_0}{\theta_1} + k_{11}\dfrac{\theta_0}{b_1h_{11} + \theta_1}e^{h_{11}x} + k_{12}\dfrac{\theta_0}{b_1h_{12} + \theta_1}e^{h_{12}x}, & x_2(1) \le x < x_2(0), \\ \pi_1, & x > x_2(0), \end{cases}$$

$$F_2(x) = \begin{cases} 0, & x < x_2(1), \\ k_{10}\dfrac{\theta_0}{\theta_2} + k_{11}\dfrac{ah_{11} + \theta_0}{\theta_2}e^{h_{11}x} + k_{12}\dfrac{ah_{12} + \theta_0}{\theta_2}e^{h_{12}x}, & x_2(1) < x \le x_2(2), \\ \pi_2, & x \ge x_2(2). \end{cases}$$

The density functions are as follows

$$f_0(x) = k_{11}h_{11}e^{h_{11}x} + k_{12}h_{12}e^{h_{12}x}, \quad x_2(1) < x < x_2(0),$$
$$f_1(x) = k_{11}h_{11}\theta_0e^{h_{11}x}/(b_1h_{11} + \theta_1) + k_{12}h_{12}\theta_0e^{h_{12}x}/(b_1h_{12} + \theta_1), \quad x_2(1) < x < x_2(0),$$
$$f_2(x) = k_{11}h_{11}e^{h_{11}x}(ah_{11} + \theta_0)/\theta_2 + k_{12}h_{12}e^{h_{12}x}(ah_{12} + \theta_0)/\theta_2, \quad x_2(1) < x < x_2(2).$$

The probability mass at the discontinuity points of the distribution function are given by

$$p_0(x_2(1)) = k_{10} + k_{11}e^{h_{11}x_2(1)} + k_{12}e^{h_{12}x_2(1)},$$
$$p_1(x_2(0)) = \pi_1 - k_{10}\,\theta_0/\theta_1 - k_{11}\theta_0e^{h_{11}x}/(b_1h_{11} + \theta_1) - k_{12}\theta_0e^{h_{12}x}/(b_1h_{12} + \theta_1),$$
$$p_2(x_2(1)) = k_{10}\theta_0/\theta_2 + k_{11}e^{h_{11}x_2(1)}(ah_{11} + \theta_0)/\theta_2 + k_{12}e^{h_{12}x_2(1)}(ah_{12} + \theta_0)/\theta_2.$$

Where $a = \alpha\lambda_1 + \lambda_2 - \mu_b$, $\quad b_1 = -\mu_v$, $\quad c = \gamma\lambda_1 + q_2\lambda_2$,

$$h_{11,12} = \left(-(\theta_0/a + \theta_1/b_1 + \theta_2/c) \pm \sqrt{\Delta_1}\right)/2,$$
$$\Delta_1 = (\theta_0/a + \theta_1/b_1 + \theta_2/c)^2 - 4(\theta_0\,\theta_2/ac + \theta_0\theta_1/ab_1 + \theta_1\theta_2/b_1c),$$
$$k_{10} = \pi_0 - k_{11}e^{h_{11}x_2(0)} - k_{12}e^{h_{12}x_2(0)}, \quad k_{11} = -\pi_0\theta_0/c_{11}\theta_1 - c_{12}k_{12}/c_{11},$$
$$k_{12} = \pi_0\theta_0(\theta_1c_{11} - \theta_2c_{13})/\theta_1\theta_2(c_{12}c_{13} - c_{11}c_{14}), \quad c_{11} = \theta_0e^{h_{11}x_2(1)}/(b_1h_{11} + \theta_1) - \theta_0e^{h_{11}x_2(0)}/\theta_1,$$
$$c_{12} = \theta_0e^{h_{12}x_2(1)}/(b_1h_{12} + \theta_1) - \theta_0e^{h_{12}x_2(0)}/\theta_1, \quad c_{13} = (ah_{11} + \theta_0)e^{h_{11}x_2(2)}/\theta_2 - \theta_0\,e^{h_{11}x_2(0)}/\theta_2,$$
$$c_{14} = (ah_{12} + \theta_0)e^{h_{12}x_2(2)}/\theta_2 - \theta_0e^{h_{12}x_2(0)}/\theta_2.$$

Proof. At state 0, the fluid level increases at rate $\alpha\lambda_1 + \lambda_2 - \mu_b$ until it reaches the maximum $x_2(0)$. At state 1, the fluid flows out at rate μ_v without any inflow because of $x_k(1) < x_k(0)$, and the fluid level decreases at rate μ_v until it reaches the threshold $x_2(1)$. At state 2, the fluid level increases at rate $\gamma\lambda_1 + q_2\lambda_2$ until it reaches the maximum $x_2(2)$. The variation trend of the fluid level in the buffer at each state is shown in Figure 1.

Considering the changes of the fluid level within a sufficiently small time interval, thus we have

$$\begin{cases} F_0(t+\Delta t,x) = F_0\left(t,x-(\alpha\lambda_1+\lambda_2-\mu_b)\Delta t\right)(1-\theta_0\Delta t) + F_2\left(t,x-\lambda\Delta t\right)\theta_2\Delta t + o(\Delta t), \\ F_1(t+\Delta t,x) = F_1\left(t,x-\left(-\mu_v\right)\Delta t\right)(1-\theta_1\Delta t) + F_0\left(t,x-(\lambda-\mu_b)\Delta t\right)\theta_0\Delta t + o(\Delta t), \\ F_2(t+\Delta t,x) = F_2\left(t,x-\left(\gamma\lambda_1+q_2\lambda_2\right)\Delta t\right)(1-\theta_2\Delta t) + F_1\left(t,x-(\lambda-\mu_v)\Delta t\right)\theta_1\Delta t + o(\Delta t). \end{cases} \tag{4.1}$$

Divide both sides of each equation in (4.1) by Δt ,and take the limit of Δt to 0 obtain

$$\begin{cases} \dfrac{\partial F_0(t,x)}{\partial t} + (\alpha\lambda_1+\lambda_2-\mu_b)\dfrac{\partial F_0(t,x)}{\partial x} = -F_0(t,x)\theta_0 + F_2(t,x)\theta_2, \\ \dfrac{\partial F_1(t,x)}{\partial t} + \left(-\mu_v\right)\dfrac{\partial F_1(t,x)}{\partial x} = -F_1(t,x)\theta_1 + F_0(t,x)\theta_0, \\ \dfrac{\partial F_2(t,x)}{\partial t} + \left(\gamma\lambda_1+q_2\lambda_2\right)\dfrac{\partial F_2(t,x)}{\partial x} = -F_2(t,x)\theta_2 + F_0(t,x)\theta_0. \end{cases} \tag{4.2}$$

In steady state, the joint probability distribution functions satisfy

$$\frac{\partial F_0(t,x)}{\partial t} = \frac{\partial F_1(t,x)}{\partial t} = \frac{\partial F_2(t,x)}{\partial t} = 0.$$

Manipulating (4.2), we establish a system of linear differential equations

$$\begin{cases} \left(\alpha\lambda_1+\lambda_2-\mu_b\right)\dfrac{dF_0(x)}{dx} = -\theta_0F_0(x) + \theta_2F_2(x), \\ \left(-\mu_v\right)\dfrac{dF_1(x)}{dx} = \theta_0F_0(x) - \theta_1F_1(x), \\ \left(\gamma\lambda_1+q_2\lambda_2\right)\dfrac{dF_2(x)}{dx} = \theta_1F_1(x) - \theta_2F_2(x), \end{cases} \tag{4.3}$$

with the boundary conditions $F_0\left(x_2(0)\right) = \pi_0, F_1\left(x_2(1)\right) = 0, F_2\left(x_2(2)\right) = \pi_2$.

Combining the boundary conditions, we solve linear differential equations (4.3) to obtain the steady-state distribution of the fluid level, the probability density function, and the probability mass at the discontinuity points, then theorem 2 is proved.

Similarly, according to the analysis method and steps of theorem 2, Based on Figures 2-4, we can obtain the following results for other three cases.

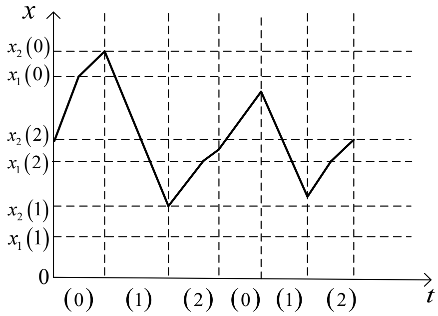

Figure 1 *Trend of Fluid Level Changes in Case 1*

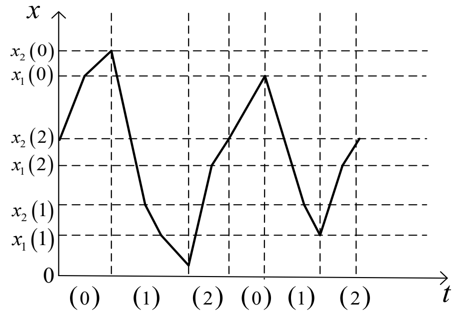

Figure 2 *Trend of Fluid Level Changes in Case 2*

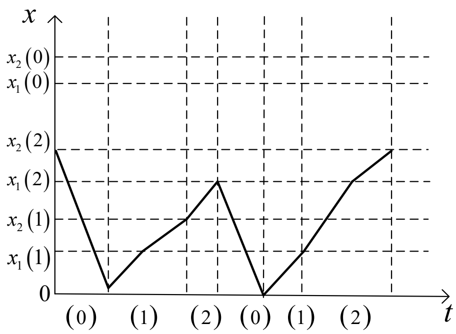

Figure 3 *Trend of Fluid Level Changes in Case 3*

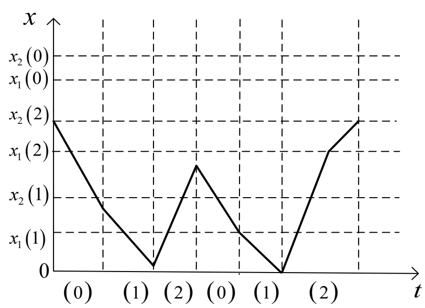

Figure 4 *Trend of Fluid Level Changes in Case 4*

Theorem 3. when $r_0 > 0, r_1 < 0$, then

$$F_0(x) = \begin{cases} 0, & x < 0, \\ k_{20} + k_{21} e^{h_{21}x} + k_{22} e^{h_{22}x}, & 0 < x \le x_2(0), \\ \pi_0, & x > x_2(0), \end{cases}$$

$$F_1(x) = \begin{cases} 0, & x < 0, \\ k_{20} \dfrac{\theta_0}{\theta_1} + k_{21} \dfrac{\theta_0}{b_2 h_{21} + \theta_1} e^{h_{21}x} + k_{22} \dfrac{\theta_0}{b_2 h_{22} + \theta_1} e^{h_{22}x}, & 0 \le x < x_2(0), \\ \pi_1, & x > x_2(0), \end{cases}$$

$$F_2(x) = \begin{cases} 0, & x < 0, \\ k_{20} \dfrac{\theta_0}{\theta_2} + k_{21} \dfrac{a h_{21} + \theta_0}{\theta_2} e^{h_{21}x} + k_{22} \dfrac{a h_{22} + \theta_0}{\theta_2} e^{h_{22}x}, & 0 < x \le x_2(2), \\ \pi_2, & x > x_2(2). \end{cases}$$

The density functions are as follows

$$f_0(x)=k_{21}h_{21}e^{h_{21}x}+k_{22}h_{22}e^{h_{22}x},\quad 0<x<x_2(0),$$
$$f_1(x)=k_{21}h_{21}\theta_0 e^{h_{21}x}\big/\left(b_2h_{21}+\theta_1\right)+k_{22}h_{22}\theta_0 e^{h_{22}x}\big/\left(b_2h_{22}+\theta_1\right),\quad 0<x<x_2(0),$$
$$f_2(x)=k_{21}h_{21}e^{h_{21}x}\left(ah_{21}+\theta_0\right)\big/\theta_2+k_{22}h_{22}e^{h_{22}x}\left(ah_{22}+\theta_0\right)\big/\theta_2,\quad 0<x<x_2(2).$$

The probability mass at the discontinuity points of the distribution function are given by

$$p_0(0)=k_{20}+k_{21}+k_{22},$$
$$p_1\left(x_2(0)\right)=\pi_1-k_{20}\,\theta_0/\theta_1-k_{21}\theta_0 e^{h_{21}x_2(0)}\big/\left(b_2h_{21}+\theta_1\right)-k_{22}\theta_0 e^{h_{22}x_2(0)}\big/\left(b_2h_{22}+\theta_1\right),$$
$$p_2(0)=k_{20}\theta_0/\theta_2+k_{21}\left(ah_{21}+\theta_0\right)\big/\theta_2+k_{22}\left(ah_{22}+\theta_0\right)\big/\theta_2.$$

Where $b_2=\omega_1(-\mu_v)+\left(1-\omega_1\right)\left(\beta\lambda_1+q_1\lambda_2-\mu_v\right)$,

$$\Delta_2=\left(\theta_0/a+\theta_1/b_2+\theta_2/c\right)^2-4\left(\theta_0\,\theta_2/a\,c+\theta_0\theta_1/ab_2+\theta_1\theta_2/b_2\,c\right),$$
$$h_{21,22}=\left(-\left(\theta_0/a+\theta_1/b_2+\theta_2/c\right)\pm\sqrt{\Delta_2}\right)\big/2,\quad k_{20}=\pi_0-k_{21}e^{h_{21}x_2(0)}-k_{22}e^{h_{22}x_2(0)},$$
$$k_{21}=\left(-\pi_0\theta_0/\theta_1-c_{22}k_{22}\right)/c_{21},\quad k_{22}=\left(c_{21}\pi_2-\pi_0\theta_0\left(c_{21}/\theta_2-c_{23}/\theta_1\right)\right)\big/\left(c_{21}c_{24}-c_{22}c_{23}\right),$$
$$c_{21}=\theta_0\big/\left(b_2h_{21}+\theta_1\right)-\theta_0 e^{h_{21}x_2(0)}\big/\theta_1,\quad c_{22}=\theta_0\big/\left(b_2h_{22}+\theta_1\right)-\theta_0 e^{h_{22}x_2(0)}\big/\theta_1,$$
$$c_{23}=\left(ah_{21}+\theta_0\right)e^{h_{21}x_2(2)}\big/\theta_2-\theta_0 e^{h_{21}x_2(0)}\big/\theta_2,\quad c_{24}=\left(ah_{22}+\theta_0\right)e^{h_{22}x_2(2)}\big/\theta_2-\theta_0 e^{h_{22}x_2(0)}\big/\theta_2.$$

Theorem 4. when $r_0<0, r_1>0$, then

$$F_0(x)=\begin{cases}0, & x<0,\\ k_{30}+k_{31}e^{h_{31}x}+k_{32}e^{h_{32}x}, & 0\le x<x_2(2),\\ \pi_0, & x>x_2(2),\end{cases}$$

$$F_1(x)=\begin{cases}0, & x<0,\\ k_{30}\dfrac{\theta_0}{\theta_1}+k_{31}\dfrac{\theta_0}{b_3h_{31}+\theta_1}e^{h_{31}x}+k_{32}\dfrac{\theta_0}{b_3h_{32}+\theta_1}e^{h_{32}x}, & 0<x\le x_2(1),\\ \pi_1, & x>x_2(1),\end{cases}$$

$$F_2(x)=\begin{cases}0, & x<0,\\ k_{30}\dfrac{\theta_0}{\theta_2}+k_{31}\dfrac{ah_{31}+\theta_0}{\theta_2}e^{h_{31}x}+k_{32}\dfrac{ah_{32}+\theta_0}{\theta_2}e^{h_{32}x}, & 0<x\le x_2(2),\\ \pi_2, & x>x_2(2).\end{cases}$$

The density functions are as follows

$$f_0(x)=k_{31}h_{31}e^{h_{31}x}+k_{32}h_{32}e^{h_{32}x},\quad 0<x<x_2(2),$$
$$f_1(x)=k_{31}h_{31}\,\theta_0 e^{h_{31}x}\big/\left(b_3h_{31}+\theta_1\right)+k_{32}h_{32}\theta_0 e^{h_{32}x}\big/\left(b_3h_{32}+\theta_1\right),\quad 0<x<x_2(1),$$
$$f_2(x)=k_{31}h_{31}\left(ah_{31}+\theta_0\right)e^{h_{31}x}\big/\theta_2+k_{32}h_{32}\left(ah_{32}+\theta_0\right)e^{h_{32}x}\big/\theta_2,\quad 0<x<x_2(2).$$

The probability mass at the discontinuity points of the distribution function are given by

$$p_0(x_2(2)) = \pi_0 - k_{30} - k_{31}e^{h_{31}x_2(2)} - k_{32}e^{h_{32}x_2(2)},$$
$$p_1(0) = k_{30}\theta_0/\theta_1 + k_{31}\theta_0/(b_3h_{31}+\theta_1) + k_{32}\theta_0/(b_3h_{32}+\theta_1),$$
$$p_2(0) = k_{30}\theta_0/\theta_2 + k_{31}(ah_{31}+\theta_0)/\theta_2 + k_{32}(ah_{32}+\theta_0)/\theta_2.$$

Where $b_3 = \beta\lambda_1 + q_1\lambda_2 - \mu_v$,

$$\Delta_3 = (\theta_0/a + \theta_1/b_3 + \theta_2/c)^2 - 4(\theta_0\,\theta_2/a\,c + \theta_0\theta_1/ab_3 + \theta_1\theta_2/b_3\,c),$$
$$h_{31,32} = \left(-(\theta_0/a + \theta_1/b_3 + \theta_2/c) \pm \sqrt{\Delta_3}\right)/2, \quad k_{30} = -k_{31} - k_{32}, \quad k_{31} = (\pi_1 - c_{32}k_{32})/c_{31},$$
$$k_{32} = (c_{31}\pi_2 - c_{33}\pi_1)/(c_{31}c_{34} - c_{32}c_{33}), \quad c_{31} = \theta_0 e^{h_{31}x_2(1)}/(b_3h_{31}+\theta_1) - \theta_0/\theta_1,$$
$$c_{32} = \theta_0 e^{h_{32}x_2(1)}/(b_3h_{32}+\theta_1) - \theta_0/\theta_1, \quad c_{33} = (ah_{31}+\theta_0)e^{h_{31}x_2(2)}/\theta_2 - \theta_0/\theta_2,$$
$$c_{34} = (ah_{32}+\theta_0)e^{h_{32}x_2(2)}/\theta_2 - \theta_0/\theta_2.$$

Theorem 5. when $r_0 < 0, r_1 < 0,$ then

$$F_0(x) = \begin{cases} 0, & x < 0, \\ k_{40} + k_{41}e^{h_{41}x} + k_{42}e^{h_{42}x}, & 0 < x \le x_2(2), \\ \pi_0, & x > x_2(2), \end{cases}$$

$$F_1(x) = \begin{cases} 0, & x < 0, \\ k_{40}\dfrac{\theta_0}{\theta_1} + k_{41}\dfrac{\theta_0}{b_4h_{41}+\theta_1}e^{h_{41}x} + k_{42}\dfrac{\theta_0}{b_4h_{42}+\theta_1}e^{h_{42}x}, & 0 \le x < x_2(2), \\ \pi_1, & x > x_2(2), \end{cases}$$

$$F_2(x) = \begin{cases} 0, & x < 0, \\ k_{40}\dfrac{\theta_0}{\theta_2} + k_{41}\dfrac{ah_{41}+\theta_0}{\theta_2}e^{h_{41}x} + k_{42}\dfrac{ah_{42}+\theta_0}{\theta_2}e^{h_{42}x}, & 0 < x \le x_2(2), \\ \pi_2, & x > x_2(2). \end{cases}$$

The density functions are as follows

$$f_0(x) = k_{41}h_{41}e^{h_{41}x} + k_{42}h_{42}e^{h_{42}x}, \quad 0 < x < x_2(2),$$
$$f_1(x) = k_{41}h_{41}\theta_0 e^{h_{41}x}/(b_3h_{41}+\theta_1) + k_{42}h_{42}\theta_0 e^{h_{42}x}/(b_3h_{42}+\theta_1), \quad 0 < x < x_2(2),$$
$$f_2(x) = k_{41}h_{41}(ah_{41}+\theta_0)e^{h_{41}x}/\theta_2 + k_{42}h_{42}(ah_{42}+\theta_0)e^{h_{42}x}/\theta_2, \quad 0 < x < x_2(2).$$

The probability mass at the discontinuity points of the distribution function are given by

$$p_0(0) = k_{40} + k_{41} + k_{42},$$
$$p_1(x_2(2)) = \pi_1 - k_{40}\,\theta_0/\theta_1 - k_{41}\theta_0 e^{h_{41}x_2(2)}/(b_3h_{41}+\theta_1) - k_{32}\theta_0 e^{h_{42}x_2(2)}/(b_3h_{42}+\theta_1),$$
$$p_2(0) = k_{40}\theta_0/\theta_2 + k_{41}(ah_{41}+\theta_0)/\theta_2 + k_{42}(ah_{42}+\theta_0)/\theta_2.$$

Where $b_4 = \omega_2(-\mu_v) + (1-\omega_2)(\beta\lambda_1 + q_1\lambda_2 - \mu_v)$

$$\Delta_4 = (\theta_0/a + \theta_1/b_4 + \theta_2/c)^2 - 4(\theta_0\theta_2/ac + \theta_0\theta_1/ab_4 + \theta_1\theta_2/b_4 c),$$

$$h_{41,42} = \left(-(\theta_0/a + \theta_1/b_4 + \theta_1/c) \pm \sqrt{\Delta_4}\right)/2, \quad k_{40} = \pi_0 - k_{41}e^{h_{41}x_2(2)} - k_{42}e^{h_{42}x_2(2)},$$

$$k_{41} = (-\pi_0\theta_0/\theta_1 - c_{42}k_{42})/c_{41}, \quad k_{42} = \left(c_{41}\pi_2 - \pi_0\theta_0(c_{41}/\theta_2 - c_{43}/\theta_1)\right)/(c_{41}c_{44} - c_{42}c_{43}),$$

$$c_{41} = \theta_0/(b_4h_{41} + \theta_1) - \theta_0 e^{h_{41}x_2(2)}/\theta_1, \quad c_{42} = \theta_0/(b_4h_{42} + \theta_1) - \theta_0 e^{h_{42}x_2(2)}/\theta_1,$$

$$c_{43} = (ah_{41} + \theta_0)e^{h_{41}x_2(2)}/\theta_2 - \theta_0 e^{h_{41}x_2(2)}/\theta_2, \quad c_{44} = (ah_{42} + \theta_0)e^{h_{42}x_2(2)}/\theta_2 - \theta_0 e^{h_{42}x_2(2)}/\theta_2.$$

4.2 Average fluid level

Denote the steady-state probability distribution of the fluid level in the buffer and the corresponding LST as follows

$$F(x) = P\{X(t) \le x\} = \sum_{i=0}^{2} F_i(x). \quad f^*(s) = \int_0^\infty e^{-sx} dF(x), \quad s > 0.$$

In situation $x_1(1) < x_2(1) < x_1(2) < x_2(2) < x_1(0) < x_2(0)$, according to theorem 2, we derive the steady-state probability distribution and average fluid level in the buffer for the following four cases.

Case 1. when $r_0 > 0, r_1 > 0$, the steady-state probability distribution of the fluid level is

$$F(x) = \begin{cases} 0, & x < 0, \\ A_0k_{10} + A_{11}k_{11}e^{h_{11}x} + A_{12}k_{12}e^{h_{12}x}, & 0 \le x \le x_2(0), \\ 1, & x \ge x_2(0), \end{cases}$$

where $A_0 = 1 + \theta_0/\theta_1 + \theta_0/\theta_2$, $A_{11} = 1 + \theta_0/(b_1h_{11} + \theta_1) + (ah_{11} + \theta_0)/\theta_2$
$A_{12} = 1 + \theta_0/(b_1h_{12} + \theta_1) + (ah_{12} + \theta_0)/\theta_2$.

The corresponding LST is

$$f^*(s) = \left(A_{11}k_{11}h_{11}/(h_{11}-s)\right)\left(e^{(h_{11}-s)x_2(0)} - 1\right) + \left(A_{12}k_{12}h_{12}/(h_{12}-s)\right)\left(e^{(h_{12}-s)x_2(0)} - 1\right).$$

The average value of the fluid level is

$$E(X) = -\lim_{s\to 0}\frac{df^*(s)}{ds}$$
$$= (A_{11}k_{11}/h_{11})\left(e^{h_{11}x_2(0)} - 1\right) + A_{11}k_{11}x_2(0)e^{h_{11}x_2(0)} + (A_{12}k_{12}/h_{12})\left(e^{h_{12}x_2(0)} - 1\right) + A_{12}k_{12}x_2(0)e^{h_{12}x_2(0)}.$$

Case 2. when $r_0 > 0, r_1 < 0$, the steady-state probability distribution of the fluid level is

$$F(x) = \begin{cases} 0, & x < 0, \\ A_0k_{20} + A_{21}k_{21}e^{h_{21}x} + A_{22}k_{22}e^{h_{22}x}, & 0 \le x \le x_2(0), \\ 1, & x \ge x_2(0), \end{cases}$$

where $A_{21} = 1 + \theta_0/(b_2h_{21} + \theta_1) + (ah_{21} + \theta_0)/\theta_2$, $A_{22} = 1 + \theta_0/(b_2h_{22} + \theta_1) + (ah_{22} + \theta_0)/\theta_2$.

The corresponding LST is

$$f^*(s)=\left(A_{21}k_{21}h_{21}/(h_{21}-s)\right)\left(e^{(h_{21}-s)x_2(0)}-1\right)+\left(A_{22}k_{22}h_{22}/(h_{22}-s)\right)\left(e^{(h_{22}-s)x_2(0)}-1\right).$$

The average value of the fluid level is

$$E(X)=\left(A_{21}k_{21}/h_{21}\right)\left(e^{h_{21}x_2(0)}-1\right)+A_{21}k_{21}x_2(0)e^{h_{21}x_2(0)}+\left(A_{22}k_{22}/h_{22}\right)\left(e^{h_{22}x_2(0)}-1\right)+A_{22}k_{22}x_2(0)e^{h_{22}x_2(0)}.$$

Case 3. when $r_0<0, r_1>0$, the steady-state probability distribution of the fluid level is

$$F(x)=\begin{cases}0, & x<0,\\ A_0k_{30}+A_{31}k_{31}e^{h_{31}x}+A_{32}k_{32}e^{h_{32}x}, & 0\le x\le x_2(2),\\ 1, & x\ge x_2(2),\end{cases}$$

where $A_{31}=1+\theta_0/(b_3h_{31}+\theta_1)+(ah_{31}+\theta_0)/\theta_2$, $A_{32}=1+\theta_0/(b_3h_{32}+\theta_1)+(ah_{32}+\theta_0)/\theta_2$.

The corresponding LST is

$$f^*(s)=A_{31}k_{31}h_{31}\left(e^{(h_{31}-s)x_2(2)}-1\right)/(h_{31}-s)+A_{32}k_{32}h_{32}\left(e^{(h_{32}-s)x_2(2)}-1\right)/(h_{32}-s).$$

The average value of the fluid level is

$$E(X)=A_{31}k_{31}\left(e^{h_{31}x_2(2)}-1\right)/h_{31}+A_{31}k_{31}x_2(2)e^{h_{31}x_2(2)}+A_{32}k_{32}\left(e^{h_{32}x_2(2)}-1\right)/h_{32}+A_{32}k_{32}x_2(2)e^{h_{32}x_2(2)}.$$

Case 4. when $r_0<0, r_1<0$, the steady-state probability distribution of the fluid level is

$$F(x)=\begin{cases}0, & x<0,\\ A_0k_{40}+A_{41}k_{41}e^{h_{41}x}+A_{42}k_{42}e^{h_{42}x}, & 0\le x\le x_2(2),\\ 1, & x\ge x_2(2),\end{cases}$$

where $A_{41}=1+\theta_0/(b_4h_{41}+\theta_1)+(ah_{41}+\theta_0)/\theta_2$, $A_{42}=1+\theta_0/(b_4h_{42}+\theta_1)+(ah_{42}+\theta_0)/\theta_2$.

The corresponding LST is

$$f^*(s)=A_{41}k_{41}h_{41}\left(e^{(h_{41}-s)x_2(2)}-1\right)/(h_{41}-s)+A_{42}k_{42}h_{42}\left(e^{(h_{42}-s)x_2(2)}-1\right)/(h_{42}-s).$$

The average value of the fluid level is

$$E(X)=A_{41}k_{41}\left(e^{h_{41}x_2(2)}-1\right)/h_{41}+A_{41}k_{41}x_2(2)e^{h_{41}x_2(2)}+A_{42}k_{42}\left(e^{h_{42}x_2(2)}-1\right)/h_{42}+A_{42}k_{42}x_2(2)e^{h_{42}x_2(2)}.$$

4.3 Average social profits per unit time

Assume that type k fluid follows the social optimal strategy $\left(x_k(0),x_k(1),x_k(2)\right)$, $k=1,2$, the average social profits per unit time for the group society can be defined as

$$S=\sum_{k=1}\left(\lambda_{ek}R_k-C_kE\left(X_k\right)\right) \tag{4.4}$$

where λ_{ek} represents the effective inflow rate of the type k fluid, and $E\left(X_k\right)$ represents the average fluid level of the type k fluid.

According to theorem 2, the following results in four cases can be obtained.

Case 1. when $r_0 > 0, r_1 > 0$, the effective inflow rates for two types of fluid are given by

$$\begin{aligned}\lambda_{ek} &= \lambda_k\left(\int_{x_2(1)}^{x_2(0)}\left(f_0(x)+f_1(x)\right)dx+\int_{x_2(1)}^{x_2(2)}f_2\left(x\right)dx+p_0\left(x_2(1)\right)+p_2\left(x_2(1)\right)\right)\\ &= \lambda_k\left(k_{10}(1+\theta_0/\theta_2)+\left(1+(ah_{11}+\theta_0)/\theta_2\right)k_{11}e^{h_{11}x_2(1)}+\left(1+(ah_{12}+\theta_0)/\theta_2\right)k_{12}e^{h_{12}x_2(1)}\right.\\ &+k_{11}\left(1+\theta_0/(b_1h_{11}+\theta_1)\right)(e^{h_{11}x_2(0)}-e^{h_{11}x_2(1)})+k_{12}\left(1+\theta_0/(b_1h_{12}+\theta_1)\right)(e^{h_{12}x_2(0)}-e^{h_{12}x_2(1)})\\ &\left.+\left(k_{11}(ah_{11}+\theta_0)/\theta_2\right)(e^{h_{11}x_2(2)}-e^{h_{11}x_2(1)})+\left(k_{12}(ah_{12}+\theta_0)/\theta_2\right)(e^{h_{12}x_2(2)}-e^{h_{12}x_2(1)})\right),\quad k=1,2\end{aligned}$$

The average fluid level for two types of fluid respectively are

$$E\left(X_1\right)=\left(\lambda_1/(\lambda_1+\lambda_2)\right)E\left(X\right),\quad E\left(X_2\right)=\left(\lambda_2/(\lambda_1+\lambda_2)\right)E\left(X\right).$$

Case 2. when $r_0 > 0, r_1 < 0$, the effective inflow rates for two types of fluid are given by

$$\begin{aligned}\lambda_{ek} &= \lambda_k\left(\int_0^{x_2(0)}\left(f_0(x)+f_1(x)+f_2\left(x\right)\right)dx+p_0(0)+p_2(0)\right)\\ &= \lambda_k\left(k_{20}(1+\theta_0/\theta_2)+k_{21}\left(1+(ah_{21}+\theta_0)/\theta_2\right)+k_{22}\left(1+(ah_{22}+\theta_0)/\theta_2\right)\right.\\ &+k_{21}\left(1+\theta_0/(b_2h_{21}+\theta_1)+(ah_{21}+\theta_0)/\theta_2\right)(e^{h_{21}x_2(0)}-1)\\ &\left.+k_{22}\left(1+\theta_0/(b_2h_{22}+\theta_1)+(ah_{22}+\theta_0)/\theta_2\right)(e^{h_{22}x_2(0)}-1)\right),\quad k=1,2\end{aligned}$$

The average fluid level for two types of fluid respectively are

$$E\left(X_1\right)=\left(\lambda_1/(\lambda_1+\lambda_2)\right)E\left(X\right),\quad E\left(X_2\right)=\left(\lambda_1/(\lambda_1+\lambda_2)\right)E\left(X\right).$$

Case 3. when $r_0 < 0, r_1 > 0$, the effective inflow rates for two types of fluid are given by

$$\begin{aligned}\lambda_{ek} &= \lambda_k\left(\pi_0+\int_0^{x_2(1)}f_1\left(x\right)dx+\int_0^{x_2(2)}f_2(x)dx+p_0\left(x_2\left(2\right)\right)+p_1\left(0\right)+p_2\left(0\right)\right)\\ &= \lambda_k\left(2\pi_0+k_{30}(\theta_0/\theta_1+\theta_0/\theta_2-1)+k_{31}\left(\theta_0/(b_3h_{31}+\theta_1)+(ah_{31}+\theta_0)\big/\theta_2-e^{h_{31}x_2(2)}\right)\right.\\ &+k_{32}\left(\theta_0/(b_3h_{32}+\theta_1)+(ah_{32}+\theta_0)\big/\theta_2-e^{h_{32}x_2(2)}\right)+\left(k_{31}\,\theta_0/(b_3h_{31}+\theta_1)\right)(e^{h_{31}x_2(1)}-1)\\ &+\left(k_{32}\,\theta_0/(b_3h_{32}+\theta_1)\right)(e^{h_{32}x_2(1)}-1))+\left(k_{31}(ah_{31}+\theta_0)/\theta_2\right)(e^{h_{31}x_2(2)}-1)\\ &\left.+\left(k_{32}(ah_{32}+\theta_0)/\theta_2\right)(e^{h_{32}x_2(2)}-1)\right),\quad k=1,2\end{aligned}$$

The average fluid level for two types of fluid respectively are

$$E\left(X_1\right)=\left(\lambda_1/(\lambda_1+\lambda_2)\right)E\left(X\right),\quad E\left(X_2\right)=\left(\lambda_1/(\lambda_1+\lambda_2)\right)E\left(X\right).$$

Case 4. when $r_0 < 0, r_1 < 0$, the effective inflow rates for two types of fluid are given by

$$\begin{aligned}\lambda_{ek} &= \lambda_k\left(\pi_0 + \int_0^{x_2(2)}\left(f_1(x)+f_2(x)\right)dx + p_0(0) + p_2(0)\right)\\ &= \lambda_k\left(\pi_0 + k_{40}\left(1+\theta_0/\theta_2\right) + k_{41}\left(1+(ah_{41}+\theta_0)/\theta_2\right) + k_{42}\left(1+(ah_{42}+\theta_0)/\theta_2\right)\right.\\ &\quad + k_{41}\left(\theta_0/(b_4h_{41}+\theta_1) + (ah_{41}+\theta_0)/\theta_2\right)(e^{h_{41}x_2(2)}-1)\\ &\quad \left.+ k_{42}\left(\theta_0/(b_4h_{42}+\theta_1) + (ah_{42}+\theta_0)/\theta_2\right)(e^{h_{42}x_2(2)}-1)\right), \quad k=1,2\end{aligned}$$

The average fluid level for two types of fluid respectively are

$$E(X_1) = \left(\lambda_1/(\lambda_1+\lambda_2)\right)E(X), \quad E(X_2) = \left(\lambda_1/(\lambda_1+\lambda_2)\right)E(X).$$

Substituting the expressions for λ_{ek} and $E(X_k)$ in four cases into (4.4) , we obtain the average social profits per unit time for the service system under the socially optimal strategy.

4.4 The thresholds of social optimal strategy

In this section, in order to achieve the maximum social benefit, we will discuss the existence and non-uniqueness of the socially optimal thresholds $\left(x_k(0), x_k(1), x_k(2)\right)$ in case 1 when $r_0 > 0, r_1 > 0$. The other three cases can be analyzed similarly.

In case 1, the expression of the social profits function is dependent on thresholds $x_2(0), x_2(1), x_2(2)$. By taking the partial derivative of the social profits function with respect to $x_2(0), x_2(1), x_2(2)$ respectively and solving the resulting partial differential equation, we can obtain the following set of stationary points of the social profits function at state 0, 1, and 2.

$$\left\{x_2^*(0)\left|\frac{\partial S\left(x_2^*(0), x_2(1), x_2(2)\right)}{\partial x_2^*(0)} = 0\right.\right\}, \left\{x_2^*(1)\left|\frac{\partial S\left(x_2(0), x_2^*(1), x_2(2)\right)}{\partial x_2^*(1)} = 0\right.\right\}, \left\{x_2^*(2)\left|\frac{\partial S\left(x_2(0), x_2(1), x_2^*(2)\right)}{\partial x_2^*(2)} = 0\right.\right\}.$$

By comparing the function values at the stationary points with the function values at the interval endpoints, we can determine the maximum $S_{\max 1}$ of the social profits function. If the fluid thresholds exist, that are $\left(x_2(0), x_2(1), x_2(2)\right) = \arg S_{\max 1}$.

5. Numerical analysis

5.1 Sensitivity analysis

This section analyzes the influence of parameters in the fluid model on the average fluid level. Considering the complexity of the function formula, Software is chosen to simulate the influence of parameter changes, take case 1 as an example.

The warehouse where AGV operates is considered as a buffer in this context. The arrival rate of normal orders is denoted as λ_1, while the arrival rate of orders for fragile goods is denoted as λ_2. μ_b and μ_v respectively represents the service rate of AGV in high and low-speed modes. θ_0, θ_1 and θ_2 are used to indicate the distribution parameters of the service system in working period, working vacation period, and vacation period respectively.

Assuming that $R_1 = 30, R_2 = 20, C_1 = C_2 = 1, \alpha = 0.8, \beta = 0.7, \gamma = 0.8, \theta_0 = \theta_1 = 1, \theta_2 = 3, q_1 = 0.6,$ $q_2 = 0.6, x_1(1) = 5, x_2(1) = 6, x_1(2) = 7, x_2(2) = 8, x_1(0) = 9, x_2(0) = 10$, the average fluid level varies with the service rate as depicted in Figure 5. The average fluid level decreases as the service rate μ_b increases, and increases as the service rate μ_v increases.

Assigning values to the parameters, $R_1 = 30, R_2 = 20, C_1 = 1, C_2 = 1, \alpha = 0.8,\ \beta = 0.7, \gamma = 0.8,$ $\theta_2 = 1, \lambda_1 = 5, \lambda_2 = 2, q_1 = 0.6, q_2 = 0.6, \mu_b = 4, \mu_v = 3, x_1(1) = 5, x_2(1) = 6, x_1(2) = 7, x_2(2) = 8, x_1(0) = 9,$ $x_2(0) = 10$, the average fluid level changes as shown in the tr 6. θ_0 has no significant effect on the average fluid level, but the fluid level tends to decrease as the θ_1 increases.

5.2 Numerical simulation

Due to the complexity of the expression for the social profits function, this section utilizes software to simulate the social benefit. Taking case 1 as example, by visualizing the simulated profits function, the impact of parameters on social benefit is discussed.

Assuming that $R_1 = 30, R_2 = 20, C_1 = C_2 = 1, \alpha = 0.8, \beta = 0.7, \gamma = 0.8, \theta_0 = \theta_1 = 1, \theta_2 = 3, q_1 = 0.6,$

$q_2 = 0.6, x_1(1) = 5, x_2(1) = 6, x_1(2) = 7, x_2(2) = 8, x_1(0) = 9, x_2(0) = 10$, the social profits varies with the service rate as depicted in Figure 7. In this case, the social profits decreases as the service rate μ_v increases during working periods.

Assigning values to the parameters, $R_1 = 30, R_2 = 20, C_1 = 1, C_2 = 1, \alpha = 0.8,\ \beta = 0.7, \gamma = 0.8,$ $\theta_2 = 1, \lambda_1 = 5, \lambda_2 = 2, q_1 = 0.6, q_2 = 0.6, \mu_b = 4, \mu_v = 3, x_1(1) = 5, x_2(1) = 6, x_1(2) = 7, x_2(2) = 8, x_1(0) = 9,$ $x_2(0) = 10$, it is observed from Figure 8 that the social profits decreases as the working period rate θ_0 and the working vacation rate θ_1 increases. Therefore, it is advisable for system managers to consider reducing the rate θ_0 of working period and the rate θ_1 of working vacation to optimize the social profits.

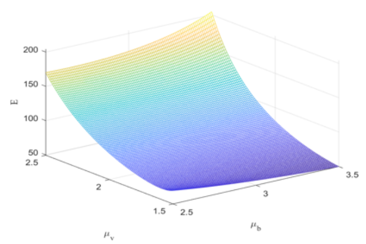

Figure 5 *Impact of μ_v, μ_b on $E(X)$ in Case 1.*

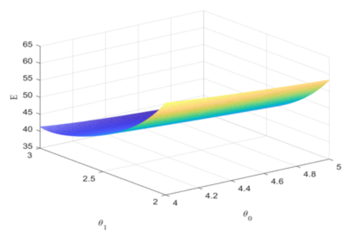

Figure 6 *Impact of θ_0, θ_1 on $E(X)$ in Case 1.*

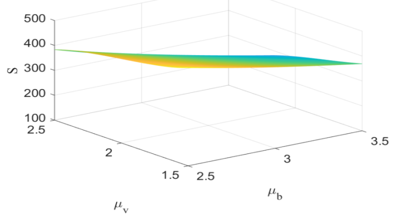

Figure 7 *Impact of μ_v, μ_b on Social Profits in Case 1.*

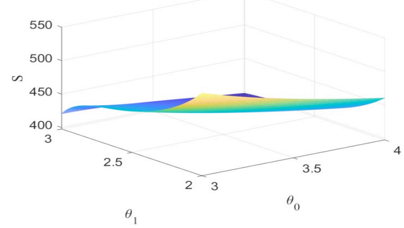

Figure 6 *Impact of θ_0, θ_1 on Social Profits in Case 1.*

5.3 Optimization algorithm

Seagull Optimization Algorithm (SOA) is a new swarm intelligent optimization algorithm proposed in recent years. It iteratively searches for the optimal value by simulating the migration and foraging behavior of seagulls in nature. The algorithm has the advantages of strong adaptability, simple structure and easy implementation. Through experimental comparison, SOA is more competitive than other related optimization algorithms in terms of convergence and computational complexity. This section uses SOA to find an approximate value for the optimal social entry threshold.

Taking case 2 as an example, the specific algorithm for using SOA iteration to solve the social optimal entry threshold $x_2(0)$ and social optimal profits S is as follows.

Step1： Define initial values for system parameters, seagull population pop, maximum number Max_{iter} of iterations, search space dimensions dim, search space upper and lower boundaries ub, lb, control factors f_c, spiral coefficients u, v, and current number of iterations $t = 1$.

Step2： Initialize seagull population location

for $m = 1 : pop$

$l_m = rand(ub - lb) + lb;$

end

Step3： Set the current optimal position as l_1 and calculate the fitness value

$l^* = l_1;$

$fitness = S(l^*)$

Step4： Simulate seagull migration and foraging behavior to update seagull positions

for $m = 1 : pop$

$A = f_c - (\frac{f_c}{Max_{iter}});$

$C_s(m) = (f_c - (\frac{f_c}{Max_{iter}}))l_m;$

$M_s(m) = 2(l^* - l_m)A^2 rand;$

$D_s(m) = |C_s(m) + M_s(m)|;$

$l_m = D_s(m)(ue^{wv})^3 w\cos w\sin w + l^*;$

% $rand$ inter random number $(0,1)$, w inter random number $(0,2\pi)$

end

Step5： Calculate fitness value

for $m = 1 : pop$

$S_m = S(l_m);$

end

Step6：Find the current optimal position l^* and fitness value

for $m = 1 : pop$

if $S_m > fitness$;

$l^* = l_m$;

$fitness = S_m$;

end

Step7：Update Iterations

$t = t + 1$;

if $t < Max_{iter}$

return Step4;

end

Step8：Output optimal position l^* and fitness value.

Assuming that $f_c = 2$, $u = 1, v = 1$, $R_1 = 100, R_2 = 50, C_1 = 1, C_2 = 1, \alpha = 0.9$, $\beta = 0.9, \gamma = 0.9$, $\lambda_1 = 2, \lambda_2 = 3, q_1 = 0.8, q_2 = 0.9, \mu_v = 1, \theta_0 = 2, \theta_1 = 2, \theta_2 = 3, x_1(1) = 1, x_2(1) = 2, x_1(2) = 2.5, x_2(2) = 3$, $x_1(0) = 4, x_1 = 8$. Based on the SOA algorithm to conduct numerical experiments to obtain a social optimal threshold of 4.9 and a social optimal benefit of 357.5578.

6. Conclusion

This paper constructs a new type of vacation fluid queuing model and analyzes from the view of economics. The mean sojourn time and stationary distribution of fluid level is derived, then the individual equilibrium balking strategy and the average social benefit per unit time are obtained by solving the utility functions. Sensitivity analysis is presented to demonstrates the effect of parameters on the fluid level and average social benefit per unit time. SOA algorithms is designed to find optimal thresholds of social benefit. The results of this paper can be applied to logistics management of OFC for e-commerce enterprises. In this paper, the equilibrium analysis of the fluid model here is carried out only in the observable case, and the equilibrium and optimization in other cases can be further discussed in the future.

Acknowledgments

This work was supported by the National Natural Science Foundation [grant number 62171143] and Natural Science Foundation Item of Hebei Province [grant number G2024203008].

Disclosure statement

No potential conflict of interest was reported by the authors.

References

Stern, T., & Elwalid, A. (1991). Analysis of separable Markov-modulated rate models for information handing systems. *Advances in Applied Probability, 23*(01), 105-139. https://doi.org/10.2307/1427514

Latouche, G., & Taylor, P. (2009). A stochastic fluid model for an ad hoc mobile network. *Queueing Systems, 63*(1-4), 109-129.

Ahn, S., & Ramaswami, V. (2003). Fluid flow models and queues-A connection by stochastic coupling. *Stochastic Models, 19*(3), 325-348. https://doi.org/10.1081/STM-120023564

Rajagopal, V., Kulkarni, S., & Stidham, S. (1995). Optimal flow control of a stochastic stochastic fluid-flow system. *IEEE Journal on Selected Areas in Communications, 13*(7), 1219-1228. https://doi.org/10.1109/49.414641

Maglaras, C. (2006). Revenue management for a multiclass single-server queue via a fluid model analysis. *Operations Research, 54*(5), 914-932. https://doi.org/10.1287 /opre.1060.0305

Kesselman, A., & Leonardi, S. (2012). Game-theoretic analysis of internet switching with selfish users. *Theoretical Computer Science, 45*(2), 107-116. https://doi.org/10.1016/ j.tcs.2012.05.029

Economou, A., & Manou, A. (2016). Strategic behavior in an observable fluid queue with an alternating service process. *European Journal of Operational Research, 254*(1), 148-160. http://dx.doi.org/10.1016/j.ejor.2016.03.046

Barron, Y. (2018). A threshold policy in a markov-modulated production system with server vacation, the case of continuous and batch supplies. *Advances in Applied Probability, 50*(4), 1256-1274. https://doi.org/10.1017/apr.2018.59.

Kelly, F., & Yudovina, E. (2018). A Markov model of a limit Order Book: Thresholds, Recurrence, and Trading Strategies. *Mathematics of Operations Research, 43*(1), 181-203. https://doi.org/10.17863/CAM.10182

Xu, X. L., Liu, J. P., & Zhang, Y. T. (2021). Equilibrium strategies in the congestion control fluid model with an unreliable server. *Journal of Systems Science and Mathematical Sciences, 41*(06), 1715-1728. https://doi.org/10.1007/s123510190 0473-5

Wang, S., & Xu, X. L. (2021). Equilibrium strategies of fluid model with threshold regulation and variable service rate. *Chinese Journal of Engineering Mathematics, 38*(1), 1-10. https://doi.org/10.3969/j.issn.1005-3085.2021.01.001

Liu, J., P., Xu X., L., Wang, S., & Yue, D., Q. (2020). Equilibrium analysis of the fluid model with two types of parallel customers and breakdowns. *Communications in Statistics - Theory and Methods, 50*(24), 5792-5805. https://doi.org/10.1080/ 03610926.2020.1737124

Wang, J., & Xu, X. L. (2024). Equilibrium analysis of the fluid model with two types of parallel customers and delayed repair. *Operations Research Transactions*, 1-15. http://kns.cnki.net/kcms/detail/31.1732.O1.20230522.1854.034.html

Zhang, Y. T., & Xu, X. L. (2022). Equilibrium strategies in the threshold control fluid model with vacations. *Journal of Systems Science and Mathematical Sciences, 42*(9), 2531-2554. https://doi.org/10.12341/jssms21493

Wang, S., & Xu, X. L. (2018). The balking strategies of fluid vacation model with setup time. *Acta Mathematicae Applicatae Sinica, 41*(6), 846-857.

Wang, S., & Xu, X. L. (2021). Equilibrium strategies of the fluid queue with working vacation. *Operational Research*, *21*(2), 1211-1228. https://doi.org/10.1007/ s12351-019-00473-5

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(Received November 2023; accepted May 2024)