http://ijims.ms.tku.edu.tw/main.php

International Journal of
Information and Management Sciences

34 (2023), 1-22. DOI:10.6186/IJIMS.202303_34(1).0001

Joint Pricing and Production Control Decisions for Omni-channel BOPS Retailers with Stochastic Reference Price Effects

Hui-ming Xu[1], *Yuan Li*[2] *and Yu-mei Hou*[1*]

[1]Yang-en University and [2]Inner Mongolia University for The Nationalities

Keywords

Omni-channel
Pricing-inventory management
Consumer returns
Consumer order cancellation
Stochastic reference price effects

Abstract.

This paper utilizes the consumers' stochastic reference price to analyze an omni-channel retailer's joint dynamic pricing and production management problem in which consumers can cancel their orders before payment and return the products after payment if the products don't meet their expectation. The retailer's optimal pricing and production rate are derived by maximizing the total expected profit under stochastic reference price effects over an infinite horizon in a continuous framework. The analysis shows that the optimal price and production rate are linear feedback form of the two state variables when production and inventory/shortage cost are strictly convex. Moreover, the sufficient conditions of stability and monotone convergence properties are derived for the expected steady state inventory and reference price. Finally, a set of sensitivity analysis is discussed to characterize the impacts of system parameters on the optimal decisions and some managerial insights are revealed.

1. Introduction

With the rapid development of mobile Internet and O2O e-commerce, online and offline integration channels has been accelerated, omni-channel retail has emerged. Omni-channel retail focuses on a truly integrated approach across the whole retail operation that delivers a seamless response to the consumer experience through all available shopping channels (see Rigby [28] and Saghiri et al. [30]). Under the omni-channel retail, consumers are more willing to switch between online and offline channels and exhibit higher level of satisfaction (see Wallace et al. [35]). Data also shows that the global e-commerce sales reached $1.86 trillion in 2016 and reached $3.88 trillion in 2018, with a growth rate of over 10% (see Statista [32]). Thus, more and more online and physical

*corresponding author

retailers have transformed to omni-channel. According to a report from Forrester Research, the sales model of pay-and-buy-online-pick- up-in-store (BOPS) is considered to be one of the most important omni-channel models and has become an important platform connecting retailers and consumers and an organic part of omni-channel retail (see Forrester Research 2014). Taking Suning as an example, consumers can enjoy store pickup, store appraisal and store return and exchange services at any physical store across the country after placing an order on Suning cloud platform (Peng and Huang [26]). In addition, BOPS can also bring potential additional transactions for offline stores, which is called cross selling profit (see Halzack [12]). A recent United Parcel Service (UPS) study shows that 45% of consumers will have new purchase when they pick up goods offline (see UPS [34]). The implementation of the BOPS gives full play to the offline advantages of traditional retailers, and enhances retailers' competitive advantages via offline professional services, convenience and experience advantages.

As an important practice of omni-channel retail, BOPS has been adopted by many retailers, but the status of consumers under the omni-channel model is prominent, retailers' pricing and inventory decisions are inevitably inseparable from the consideration of consumers' purchase behavior. Consumer's reference price is one of the important factors that affect their purchasing decisions. Behavioral science pointed out that consumers will remember the past price information of a product in their mind when they repeatedly purchase a product, thereby establishing the concept of "fair price", which is called reference price (see Kalyanaram and Winer [20]). When a new purchase decision is made for the same product, consumers will take the reference price as the benchmark of the current selling price of the product. If the current selling price is lower (higher) than the reference price, they will think that they have "earned" ("lost") and are therefore more likely (unwilling) to buy. Due to the heterogeneity of consumers, the dynamics of reference price thus may be subject to noise. How this disturbance of reference price affects the pricing and inventory decisions of omni-channel retailers remains to be studied. Moreover, the common phenomenon of consumer return under the BOPS mode will also affect the decision-making of retailers (see Zhang et al. [40]). Thus, how should the retailer make its pricing and inventory strategies to realize the product price advantage, save inventory cost and improve profit? These are the practical problems that need to be solved urgently when the retailer implements the BOPS mode. The solution of these problems meets the current development needs and has important practical significance.

In order to solve the above problems, this paper studies a continuous time joint pricing and production decisions of an omni-channel BOPS retailer based on the influence of consumers' stochastic reference price. By analyzing the mechanism of the impact of stochastic reference price on the pricing and production strategies, it provides theoretical suggestions for omni-channel retailers. Whereas, to our best knowledge, previous research on joint pricing and inventory decisions of omni- channel retailers mainly focused on single ordering cycle, and ignored the consumers' reference price effects (see Liu and Xu [24] and Zhang et al. [40]). Although recent studies considered the impact of reference price effects on the pricing and inventory of omni-channel retailers, the reference price are deterministic (see Li [22]). Besides, previous literature on coordination pricing and inventory control problem with reference price effects mainly focus on the single-channel supply chain, little attention has been paid to omni-channel field (see Chen et al. [4] and Gler et al. [11]). Thus, this paper tries to fill the gap in the decision-making of joint pricing and inventory with stochastic reference price effects in omni-channel environment. The differences between the existing joint pricing and inventory literature and ours are as follows: (1) The existing literature are focused on the deterministic reference price, while we consider the stochastic reference price effects. (2) The existing literature studies the periodic review joint pricing and inventory model, while we focus on continuous time joint pricing and production model. The model we studied incorporates the following features: (1) Omni-channel business mode; (2) Price and stochastic reference price sensitive demand function; (3) A continuous time, dynamic environment; (4) Two controls at each time: product sales price and production rate.

The rest of the paper is organized as follows. Section 2 reviews related literature. Section 3 presents a theoretical model to formulate the omni-channel with stochastic reference price effects. Section 4 investigates several structural properties of the optimal strategies. Numerical results of sensitivity analysis are represented in Section 5. Some managerial insights are provided in Section 6. Section 7 concludes our paper. All the proofs are presented in Appendix.

2. Literature Review

This paper is related to two streams of literature: one delves into omni-channel pricing and inventory management while the other discusses the joint pricing and inventory model with reference price effects. We review the related areas below.

The first relevant area is research related to pricing and inventory management of omni-channel BOPS retailers. First of all, in the study of the pricing decisions of retailers alone, Jin et al. [19] derived the optimal decisions on the product price and recommend service area for the retailer adopting BOPS. Zhang et al. [41] analyzed the pricing strategies when retailers who implement the BOPS model compete with those who only implement network channel operation. Harsha et al. [14] considered the pricing strategy of online and offline inventory sharing. Li et al. [23] explored an omni-channel pricing problem under which the retailer offers coupons via online channels for market share and profit. Secondly, in the aspect of supply chain coordination pricing decisions, He et al. [15] investigated the BOPS mode where the manufacturer sells products to online consumers directly and the online ordered products are delivered from the offline retailer's store. They derive the optimal pricing decisions for the manufacturer and retailer respectively and identify the optimal channel strategy from the perspectives of demand and profit. Fan et al. [7] discussed the problem of pricing and inventory decision-making where an online retailer and an offline retailer cooperate to implement the BOPS mode. Further, it worth mentioning that the BOPS mode require consumers to pay at the moment of ordering online, and they know the exact value of the product only after receiving/picking it, which may cause many product returns (see Zhang et al. [40]). Thus, Liu and Xu [24] studied an omni-channel BOPS retailer's pricing and ordering model considering online returns. Zhang et al. [40] considered the omni-channel BOPS mode that allows both online and offline consumers to return and cancel orders, where consumers do not need to pay immediately after placing an order. Li [22] extended the model in [40] to the multi-period case and considers the consumers' reference price effects. He et al. [17] considered an omni-channel pricing and inventory model where consumers can return unsatisfied products to the retailer, and the returned items can be resold as refurbished products in the circular economy era. He et al. [16] further studied centralized and decentralized models for the pricing and inventory decisions of deterioration products with a vendor and a retailer. The above literature seldom consider the consumers' reference price effects. In reality, the reference price effects has significant impact on retailers' operation and marketing strategies.

The second stream of related research is on joint pricing and inventory model with reference price effects. This line of research mainly focuses on two aspects, i.e., periodic review and continuous review. The research on periodic review joint pricing and inventory model under reference price effects started with Gimpl-Heersink [10], who proves the optimality of base-stock-list-price for the two-period model when the customers are loss neutral. The recent research on this aspect are as follows. Gler et al. [11] analyzed a multi-period joint pricing and inventory problem of a single item with stochastic demand subject to reference price effects, they use the safety stock as a decision variable to characterize the steady state solutions to the problem when the planning horizon is infinite. Wu et al. [38] considered a multi-period joint pricing and inventory model when strategic consumers are affected by reference price. Chen et al. [4] introduced a concave transformation to ensure the concaveness of profit function, and then prove the optimality of the base-stock-list-price strategy. Song et al. [31] studied a multi-period joint pricing and inventory model when loss-averse consumers are affected by reference price. Wang et al. [36] further analyzed a joint pricing and inventory control model when loss-neutral, loss-averse and gain-seeking consumers coexist. Li and Teng [21] investigated a multi-period pricing and inventory decisions for perishable goods when demand depends on selling price, reference price, product freshness and displayed stocks. Li [22] considered a multi-period pricing and inventory decisions of an omni-channel retailer. However, continuous time pricing and inventory system considering reference price effects is more recent. Zhang et al. [42] extended the above symmetric reference price to asymmetric form and study a joint dynamic pricing and production model with asymmetric reference price effects. The reference price aforementioned are deterministic, the literature on pricing and inventory considering stochastic reference price is relatively rare. In terms of periodic review, Bi et al. [2] constructed a periodic review dynamic pricing model by incorporating the concept of memory window into the stochastic reference price and then examine the optimal pricing strategies of a monopolist in response to consumers who exhibit loss aversion and reference dependence. Furthermore, in the framework of continuous review, Chen et al. [5] considered a dynamic pricing problem of a firm facing stochastic reference price effects. Wu and Wu [37] further investigated a dynamic pricing and risk analytics problem under competition and stochastic reference price effects. Cao and Duan [3] analyzed a joint pricing and production inventory system under stochastic reference price effects. For other related works in this stream of research, interested readers may refer to the recent review by Ren and Huang [27].

In summary, the following conclusions can be drawn from the above literature review. First of all, in the first stream of research, it is rare to study the pricing and inventory strategies of omni-channel BOPS retailers under the reference price effects. Although Li et al. [22] has considered this issue, the reference price is deterministic. Secondly, in the second stream of research, although existing studies have shown that the reference price effects have important impacts on retailers' pricing and inventory decisions, the reference price considered is also deterministic. Thirdly, in the research on the effects of stochastic reference price, current research mainly explores how pricing strategies should account for the stochastic reference price effects in the absence of inventory consideration (see Chen et al. [5] and Wu and Wu [37]). Although Cao and Duan [3] has explored the impacts of stochastic reference price effects on retailers' joint pricing and inventory strategies, it is considered from the single-channel perspective. This gives reason for us to fills the gap in the decision-making of joint pricing and inventory with stochastic reference price effects in omni-channel environment.

3. Model Description

Consider a retailer ("he"), initially an online retailer, who previously operated a single online channel that only served for online consumers. At present, he has added a physical store and implemented the omni-channel strategy, which allows consumers to place orders online without paying immediately. With this allowance, a consumer ("she") who places an order online can choose to (i) pay online and wait for the package delivered, or (ii) visit the physical stores to touch and feel the product before payment, and then buy and pick up it (i.e., choose "BOPS"). The buying procedure of a consumer who orders online is depicted in Figure 1, where and be the fractions of consumers who place orders online choose to (i) and (ii), respectively.

As shown in Figure 1, after placing an order online, if a consumer chooses to (i), she knows the exact value of the product only after receiving it. If the actual value doesn't meet her expectation, she can return the product with a full refund (see Choi and Guo [6]), but she needs to pay the return shipping fee for each unit of the product (see Li and Teng [21]). Suppose the forward shipping fee is paid by the retailer while the return shipping fee is paid by the consumer and the express company charges the same shipping fee m for both of them. If an online consumer chooses to (ii), she needs to visit the store and then decides whether to keep the product or cancel the order in store. The cancellation of the orders doesn't pay anything. The return probabilities of consumers who choose to (i) and (ii) are ξ and ϕ, respectively. Because offline consumers and BOPS consumers have the same perception of the product, the probability of offline consumers returning is also assumed to be ϕ. Moreover, assume that there is an additional cross selling profit l from every consumer visiting the store (see UPS [34]).

The omni-channel retailer orders and sells a single-item over an infinite planning horizon. The retailer charges the same price $p(t)$ at time t over the physical and online channels. Let $R(t)$ denote the consumer's reference price, according to Fibich et al. [8] and Zhang et al. [39], the reference price is modeled as continuous weighted average of past prices with an exponentially decaying weighting function, namely

$$R(t) = \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} p(s) ds,$$

Figure 1: Buying procedure of a consumer who orders online under the omni-channel environment.

where $\lambda > 0$ is called "memory effect" parameter ($0 \leq \lambda \leq 1$), which represents the degree of forgetfulness about the product's prior prices. A higher λ indicates that consumers are more impacted by the product's latest prices, i.e., the consumers are lower loyalty. If $\lambda = 0$ consumers never remember any past prices and the reference price $R(t)$ will remain at the initial reference price $R(0) = R_0$, which is a constant. We extend the above reference price formation using a stochastic differential equation:

$$dR(t) = \lambda(p(t) - R(t))dt + \varepsilon\sqrt{R(t)}dY(t), \tag{3.1}$$

where $Y(t)$ denotes a standard Wiener process and the square-root diffusion term $\varepsilon\sqrt{R(t)}$ represents the volatility of stochastic reference price.

The demand rate function at time t in the presence of reference price effects is

$$D(p(t), R(t)) = \beta_0 - \beta_1 p(t) + \gamma(R(t) - p(t)), \tag{3.2}$$

where β_0 denotes the basic market size, β_1 represent the effect intensity of the current sales price and γ reflects the reference price effects. A higher γ implies that consumers are more sensitive to the gap between the sales price and reference price, $\gamma = 0$ represents having no reference price effects. Then the cumulative demand up to time t, $Q(t)$, can be determined by the following stochastic process

$$dQ(t) = D(p(t), r(t)) - \delta dW(t), \quad t \geq 0, \tag{3.3}$$

where the Wiener process $W(t)$ is not correlated with $Y(t)$.

Let k be the fraction of the demand served under the single online channel, i.e., the demand was $kdQ(t)$ at time t when the retailer used to operate a single online channel. The omni-channel will lead to an incremental demand $(1-k)dQ(t)$ for the retailer from both the online and offline channels at time t. Let ρ and $1-\rho$ be the fractions of the incremental demand coming from the online and offline market, respectively. Then at time t, the online channel demand is increased from $kQ(t)$ to $[k + \rho(1-k)]dQ(t)$, and the offline channel demand is increased from 0 to $(1-\rho)(1-k)dQ(t)$.

The inventory is shared across channels (see Harsha and Subramanian [13]), the production/replenishment is instantaneous and infinite and the lead time is zero. The dynamics of inventory is influence by the production rate $u(t)$ and the cumulative demand $Q(t)$, which is given by the following stochastic differential equation:

$$dX(t) = u(t) - dQ(t). \tag{3.4}$$

Taking (3.2) and (3.3) into account, the above inventory dynamics (3.4) becomes

$$dX(t) = [u(t) - (\beta_0 - \beta_1 p(t) + \gamma(R(t) - p(t)))]dt + \delta dW(t),\ X(0) = X_0,\ t \geq 0. \tag{3.5}$$

where $X(0) = X_0$ is the initial inventory level. Moreover, let $C(u(t))$ and $H(X(t))$ be the production and inventory/shortage cost, respectively. Both of them are assumed to be twice continuously differentiable, nonnegative and strictly convex.

Based on the above analysis, the omni-channel retailer's expected total profit can be written as

$$\begin{aligned} J = E\int_0^{\infty} e^{-zt}\{[&pK_1(1-\sigma)(1-\xi)+sK_1(1-\sigma)\xi-mK_1(1-\sigma)+pK_1\sigma(1-\phi)+sK_1\sigma\phi \\ &+lK_1\sigma+pK_2(1-\phi)+sK_2\phi+lK_2]dQ(t)-C(u(t))-H(X(t))\}dt, \end{aligned} \tag{3.6}$$

where E denotes the expectation operator. $z > 0$ stands for the discount rate, which is an exogenous constant and determined by the cost of capital. $K_1 = \rho(1-k) + k$, $K_2 = (1-\rho)(1-k)$ and $K_1 + K_2 = 1$. $\lambda_1 = (1-\sigma)K_1$, $\lambda_2 = \sigma K_1 + K_2$ and $\lambda_1 + \lambda_2 = 1$. In the curly brackets of (3.6), the first three terms denote the revenue from online consumers who buy online directly at time t, while the fourth to sixth terms represent the revenue from BOPS consumers at time t. The seventh to ninth terms represent the revenue from offline consumers at time t. Moreover, (3.6) can be arranged as follows:

$$\begin{aligned} J =& E\int_0^{\infty} e^{-zt}\{[\lambda_1((p-s)(1-\xi)-m) + \lambda_2((p-s)(1-\phi)+l) + s]D(p(t), R(t)) \\ &- C(u(t)) - H(X(t))\}dt - E\int_0^{\infty} e^{-zt}p(t)\delta dW(t) \\ =& E\int_0^{\infty} e^{-zt}\{[\lambda_1((p-s)(1-\xi)-m) + \lambda_2((p-s)(1-\phi)+l) + s]D(p(t), R(t)) \\ &- C(u(t)) - H(X(t))\}dt, \end{aligned}$$

where the last equation holds since $E\int_0^{\infty} e^{-zt}p(t)\delta dW(t) = 0$ by Theorem 4.1.14(e) in Arnold [1].

The retailer's objective is to find the optimal pricing $p^*(t)$ and production rate $u^*(t)$ by maximizing the total profit J over the infinite planning horizon from the admissible control set $\Omega = \{(p(t), u(t)) \mid (p(t), u(t)) \in [\underline{p}, \overline{p}] \times [0, \infty)\}$, specified as

$$\begin{aligned} \max_{u(t),p(t)\in\Omega}\ & E\int_0^{\infty} e^{-zt}\{[\lambda_1((p-s)(1-\xi)-m) + \lambda_2((p-s)(1-\phi)+l) + s] \\ & \times[\beta_0 - \beta_1 p(t) + \gamma(R(t) - p(t))] - C(u(t)) - H(X(t))]dt \\ \text{s.t.}\quad & dX(t) = [u(t)-(\beta_0-\beta_1 p(t)+\gamma(R(t)-p(t)))]dt+\delta W(t),\ X(0) = X_0, \\ & dR(t) = \lambda(p(t) - R(t)) + \varepsilon\sqrt{R(t)}dY(t),\ R(0) = R_0. \end{aligned} \tag{3.7}$$

Table 1: Summary of notations.

Notation	Description
Decision variables	
$p(t)$	The omni-channel retailer's retail price at time t.
$u(t)$	The unit production rate at time t.
Parameters	
$C(\cdot), H(\cdot)$	The production cost and inventory holding/shortage cost, respectively.
$Y(t), W(t)$	The standard Wiener processes.
z	The discount factor ($0 < z < 1$).
γ	The magnitude of the reference price effects.
λ	The memory factor ($0 < \lambda < 1$).
c	The unit inventory and procurement cost.
m	The unit shipping fee.
t	The unit travelling cost of consumers to visit the store.
s	The salvage price for a leftover unit.
l	The cross-selling benefit.
ξ	The return rate of online channel consumers who choose to pay online and wait for product shipment.
ϕ	The return rate of BOPS and offline channel consumers.
σ	The fraction of online consumers choosing to BOPS and $1-\sigma$ is the fraction of online consumers choosing to buy online directly.
ρ	The fraction of the incremental demand (brought by the omni-channel strategy) coming from the online market and $1-\rho$ is the fraction of the incremental demand coming from offline market.
k	The fraction of the market demand served under the single online channel strategy.

The related parameters and variables used in this paper are summarized in Table 1; other notations will be defined as needed.

4. Structural Analysis of the Optimal Omni-channel Strategies

In this section, we analyze the structural properties of the omni-channel retailer's optimal pricing and production rate by solving the optimization problem (3.7). We adopt a continuous-time version of dynamic programming approach. Let $V(X, R)$ be the profit-to-go function, i.e., the optimal profit onward if the current inventory and reference price pair is (X, R). Then the optimization problem (3.7) is equivalent to solving the following HJB equation:

$$\begin{aligned} zV = \max_{u(t),p(t)\in\Omega} \Big\{ & [\lambda_1((p-s)(1-\xi)-m)+\lambda_2((p-s)(1-\phi)+l)+s] \\ & \times[\beta_0-\beta_1 p(t)+\gamma(R(t)-p(t))] - C(u(t)) - H(X(t)) \\ & + V_X[u(t)-(\beta_0-\beta_1 p(t)+\gamma(R(t)-p(t)))] \\ & + V_R\lambda(p(t)-R(t)) + \frac{1}{2}\delta^2 V_{XX} + \frac{1}{2}\varepsilon^2 R V_{RR} \Big\}, \end{aligned} \tag{4.1}$$

where $V_i = \partial V/\partial i$ and $V_{ii} = \partial^2 V/\partial i^2$, $i = X, R$. Equation (4.1) enables us to derive the feedback sales pricing $p^*(t)$ and production rate $u^*(t)$ as summarized below. For the smoothness of the paper, all the proofs of main results are available in Appendix.

Proposition 1. *The optimal sales price and production rate of the optimization problem* (3.7) *are:*

(i) *If* $\underline{p} \le \frac{(1-\lambda_1\xi-\lambda_2\phi)(\beta_0+\gamma R)+(\beta_1+\gamma)V_X+\lambda V_R-(\lambda_1\xi+\lambda_2\phi)s}{\Delta_1} \le \overline{p}$, *then the optimal sales price is*

$$p^* = \frac{(1-\lambda_1\xi-\lambda_2\phi)(\beta_0+\gamma R)+(\beta_1+\gamma)V_X+\lambda V_R-(\lambda_1\xi+\lambda_2\phi)s}{\Delta_1}, \tag{4.2}$$

where $\Delta_1 = (\beta_1+\gamma)+(1-\lambda_1\xi-\lambda_2\phi)$. *Otherwise, the solution is a boundary solution, i.e.,* $p^* = \underline{p}$ *or* $p^* = \overline{p}$.

(ii) *If* $C_u(0) < V_X$, *the optimal production rate is given by*

$$u^* = C_u^{-1}(V_X), \tag{4.3}$$

where C_u *is the partial derivative of* C *with respect to* u. *Otherwise,* $u^* = 0$.

Follows from Proposition 1 that the expressions of optimal sales price and production rate are implicit, in order to express the explicit solution of the optimal sales price and production rate conveniently, and in line with many previous production control models (see Cao and Duan [3] and Heron and Kogan [18]), all cost functions are assumed to be quadratic, which is given by

$$C_u = c_1 u + c_2 u^2 \text{ and } H(X) = x_1 X + x_2 X^2, \tag{4.4}$$

where $c_2, x_2 > 0$. Then the following result is immediately holds.

Corollary 1. *The optimal sales price and production rate of the optimization problem* (7) *are:*

(i) *If* $\underline{p} \le \frac{(1-\lambda_1\xi-\lambda_2\phi)(\beta_0+\gamma R)+(\beta_1+\gamma)V_X+\lambda V_R-(\lambda_1\xi+\lambda_2\phi)s}{\Delta_1} \le \overline{p}$, *the optimal sales price is given by*

$$p^* = \frac{(1-\lambda_1\xi-\lambda_2\phi)(\beta_0+\gamma R)+(\beta_1+\gamma)V_X+\lambda V_R-(\lambda_1\xi+\lambda_2\phi)s}{\Delta_1}, \tag{4.5}$$

where $\Delta_1 = (\beta_1+\gamma)+(1-\lambda_1\xi-\lambda_2\phi) > 0$. *Otherwise, the solution is a boundary solution, i.e.,* $p^* = \underline{p}$ *or* $p^* = \overline{p}$.

(ii) *If* $c_1 < V_X$, *the optimal production rate is given by*

$$u^* = \frac{1}{2c_2}(V_x - c_1). \tag{4.6}$$

Otherwise, $u^* = 0$.

Corollary 1 indicates that the optimal sales price and production rate depend on the partial derivatives of the omni-channel retailer's profit-to-go function $V(X, R)$. Through calculation, we give the complete expression of the profit-to-go function in Proposition 2 below, which enables us to obtain the fully expression of the optimal sales price and production rate.

Proposition 2.

(i) *The following expression satisfies the HJB equation* (4.1) *associated with problem* (3.7)*:*

$$V(X,R) = a_1 + a_2X + a_3X^2 + a_4XR + a_5R + a_6R^2, \tag{4.7}$$

where the coefficients a_i $(i = 1, 2, \ldots, 6)$ *are given in the Appendix.*

(ii) *With the quadratic cost function* (4.4) *and the coefficient* a_i $(i = 1, 2, \ldots, 6)$*, if* $\underline{p} = \frac{1}{\Delta_1}(\alpha_1 + \alpha_2X + \alpha_3R) \leq \overline{p}$*, then the optimal sales price can be expressed by:*

$$p^*(X,R) = \frac{1}{\Delta_1}(\alpha_1 + \alpha_2X + \alpha_3R), \tag{4.8}$$

where $\Delta_1 = (\beta_1 + \gamma) + (1 - \lambda_1\xi - \lambda_2\phi) > 0$*,* $\alpha_1 = (1 - \lambda_1\xi - \lambda_2\phi)\beta_0 + (\beta_1 + \gamma)a_2 + \lambda a_5 - (\lambda_1\xi + \lambda_2\phi)s$*,* $\alpha_2 = 2(\beta_1 + \gamma)a_3 + \lambda a_4$*,* $\alpha_3 = (1 - \lambda_1\xi - \lambda_2\phi)\gamma + (\beta_1 + \gamma)a_4 + 2\lambda a_6$*.*

(iii) *With the quadratic cost function* (4.4) *and the coefficient* a_i $(i = 1, 2, \ldots, 6)$*, if* $a_2 - c_1 + 2a_3X + a_4R \geq 0$*, then the optimal sales price can be expressed by:*

$$u^*(X,R) = \frac{1}{2c_2}(a_2 - c_1 + 2a_3X + a_4R). \tag{4.9}$$

Proposition 3 shows that the optimal sales price and production rate at time t depend on the current inventory level and reference price pair $(X(t), R(t))$ and both of them are linear combination of X and R. In addition, because the coefficients a_i $(i = 1, 2, \ldots, 6)$ in (4.8) and (4.9) are highly nonlinear, it is difficult to show the relation-ship between the optimal price and the optimal production rate with respect to X and R, so we will analyze it through the numerical analysis in Section 5.

Because the planning period is infinite, we will pay more attention to the optimal sales price and production rate in the steady state $(t \to \infty)$, which is shown in the following proposition.

Proposition 3. *The optimal sales price* p^*_{ss} *and the optimal production rate* u^*_{ss} *in steady state, the resulting expected steady state inventory* x_{ss}*, reference price path* r_{ss} *as well as profit can be given by:*

$$p^*_{ss} = r^*_{ss} = \frac{c_2\alpha_2[\Delta_1(2c_1\beta_0 - a_2 + c_1) - 2c_1\alpha_1(\beta_1 + \gamma)] + 2c_1\alpha_1[\Delta_1a_3 + c_2\alpha_2(\beta_1 + \gamma)]}{\alpha_2c_1[\Delta_1a_4 + 2c_2\alpha_3(\beta_1 + \gamma) - 2\Delta_1c_2\gamma] + 2c_1(\alpha_3 - \Delta_1)[\Delta_1a_3 + c_2\alpha_2(\beta_1 + \gamma)]},$$

$$u^*_{ss} = \frac{1}{2c_2}(a_2 - c_1 + 2a_3X + a_4r^*_{ss}),$$

$$x_{ss} = \frac{c_1\alpha_1[\Delta_1(a_4 - 2c_2\gamma) + 2c_2\alpha_3(\beta_1 + \gamma)] + c_2(\alpha_3 - \Delta_1)[\Delta_1(2c_2\beta_0 - a_2 + c_1) - 2c_1\alpha_1(\beta_1 + \gamma)]}{2c_1(\alpha_3 - \Delta_1)[\Delta_1a_3 + c_2\alpha_2(\beta_1 + \gamma)] - c_1\alpha_2[\Delta_1a_4 + 2c_2\alpha_3(\beta_1 + \gamma) - 2\Delta_1c_2\gamma]},$$

and

$$V_{ss} = a_1 + a_2x_{ss} + a_3(x_{ss})^2 + a_4x_{ss}r_{ss} + a_5r_{ss} + a_6(r_{ss})^2,$$

where the subscript "ss" stands for the steady state, $\Delta_1 = (\beta_1 + \gamma) + (1 - \lambda_1\xi - \lambda_2\phi) > 0$*,* $\alpha_1 = (1 - \lambda_1\xi - \lambda_2\phi)\beta_0 + (\beta_1 + \gamma)a_2 + \lambda a_5 - (\lambda_1\xi + \lambda_2\phi)s$*,* $\alpha_2 = 2(\beta_1 + \gamma)a_3 + \lambda a_4$*,* $\alpha_3 = (1 - \lambda_1\xi - \lambda_2\phi)\gamma + (\beta_1 + \gamma)a_4 + 2\lambda a_6$*. the coefficients* a_i $(i = 1, 2, \ldots, 6)$ *are given in the Appendix.*

The following proposition provides the sufficient condition for globally asymptotic stability and convergence of the expected steady state inventory x_{ss} and reference price r^*_{ss}.

Proposition 4. *For the optimization problem* (3.7), *if*

$$2(\alpha_3 - \Delta_1)[\Delta_1 a_3 + c_2\alpha_2(\beta_1 + \gamma)] - \alpha_2[\Delta_1 a_4 + 2c_2(\beta_1 + \gamma) - 2\gamma c_2\Delta_1] > 0,$$

and

$$\Delta_1 a_3 + c_2\alpha_2(\beta_1 + \gamma) + \lambda c_2(\alpha_3 - \Delta_1) < 0,$$

then the expected steady state inventory x_{ss} and reference price r^*_{ss} are asymptotically stable. Further, if $[\mathrm{tr}(J)]^2 - 4\det(J) \geq 0$, *the stable steady inventory x_{ss} and reference price r^*_{ss} are monotonically convergent, i.e., it is a sink. Otherwise, the convergence is with transient oscillations, i.e., it is a spiral sink.*

5. Numerical Analysis

In this section, we proceed several numerical experiments to illustrate the above theoretical results and gain some managerial insights. All experiments below are performed in MATLAB R2014b on a laptop with an Intel(R) Core (TM) i5-7200U central processing unit CPU (2.50GHz, 2.70GHz) and 8.0 GB of RAM running 64-bit Windows 10 Enterprise.

Consider an omni-channel inventory system with the following initial parameter values: $\beta_0 = 100$, $\beta_1 = 2$, $s = 1$, $m = 2$, $l = 1$, $\lambda = 0.2$, $\gamma = 1.25$, $z = 0.15$, $\sigma = 0.3$, $\xi = 0.04$, $\phi = 0.03$, $\rho = 0.6$, $k = 0.6$, $\varepsilon = 0.1$, $\delta = 0.75$, $\underline{p} = 20$, $\overline{p} = 100$. Moreover, the total inventory holding/shortage cost and total production cost are $-2X + 0.5X^2$ and $-5u + 5u^2$, respectively. With the parameters above, we can calculate that $a_1 = 111751.8561047051$, $a_2 = 27.8388477405554$, $a_3 = -0.628483741156556$, $a_4 = 1.544429434508493$, $a_5 = 391.114589523677$ and $a_6 = 5.0069141904786$ from the Riccati System in the Appendix. Further, the values of α_i $(i = 1, 2, 3)$ in (15) are $\alpha_1 = 265.0752930615404$, $\alpha_2 = -3.7742584306159 16$ and $\alpha_3 = 8.259811338344042$. Thus, the optimal sales price p^* and production rate u^* in (15) and (16) are

$$p^* = 62.90169550050023 - 0.8956219639250X + 1.9600323052842R, \tag{4.10}$$

and

$$u^* = 3.2838847740555 - 0.1256967482313X + 0.155442943408R. \tag{4.11}$$

It follows from (4.10) and (4.11) that both the optimal sales price p^* and production rate u^* are increasing in the consumers' reference price R and decreasing in the inventory level X. Moreover, follows from the parameter values calculated above that $\det(J) = 0.64093963306223$ and $\mathrm{tr}(J) = -2.84461669930959$. Hence, we have $[\mathrm{tr}(J)]^2 - 4\det(J) = 5.527203659455924 > 0$, this indicates that both the expected steady state of inventory level $x_{ss} = 129.7039977168851$ and reference price $r^*_{ss} = 55.481522200820145$ are global asymptotically stable and the convergence to these states is monotonic. Furthermore, the sensitivity analysis of the key system parameters, including the demand uncertainty δ, the reference price uncertainty ε, the reference price effect intensity γ, the memory parameter λ, the return rate ξ and ϕ, the fraction of BOPS σ, the shipping fee m and the cross-selling profit l, is presented on the expected steady state sales price p^*_{ss}, reference price r^*_{ss}, production rate u^*_{ss}, inventory level x_{ss} and profit V_{ss}. The corresponding computational results are presented in Tables 2-7.

As shown in Table 2, although the demand uncertainty δ does not affect the optimal sales price p^*_{ss}, production rate u^*_{ss} and inventory level x_{ss} in the steady state, but it affects the retailer's profit V_{ss}. Table 2 indicates that the retailer's profit V_{ss} decrease as demand uncertainty increases. This means that the demand uncertainty has an important impact on retailers' profitability. Especially in today's omni-channel era, the behavior of consumers becomes more uncertain, which will also affects the demand forecast of retailers. In addition, the demand forecast also needs to consider the demand of online and offline consumers. Therefore, it is particularly important for retailers to accurately predict consumer demand in the context of omni-channel era, thereby reducing the impact of demand uncertainty on their profitability. Moreover, Table 2 also presents the impact of reference price uncertainty ε on the retailer's optimal sales price p^*_{ss}, production rate u^*_{ss}, inventory level x_{ss} and profit V_{ss} in the steady state. From Table 2, we can see that the optimal sales price p^*_{ss}, inventory level x_{ss} and profit V_{ss} increase with the reference price uncertainty, while the production rate u^*_{ss} decreases. This can be intuitively illustrated as follows. The increase in the uncertainty of the reference price will cause the retailer increase the sales price, although in the short term, it will make consumers feel that the difference between reference price and actual sales price is too large, which will affect their purchase desire, thus leading to an increase in the retailer's inventory level and a decrease in production rate. However, in the long run, with the increase of consumers' reference price, the gap between reference price and actual sales price will decrease, which will eventually benefit the retailer.

Table 2: The effect of demand uncertainty δ and reference price uncertainty ε on p^*_{ss}, r^*_{ss}, u^*_{ss}, x_{ss} and V_{ss}.

ε	δ	$p^*_{ss}(r^*_{ss})$	u^*_{ss}	x_{ss}	V_{ss}
0.1	0.45	55.48152	-4.39527	129.70399	1234741.9968
0.1	0.55	55.48152	-4.39527	129.70399	1234741.5778
0.1	0.65	55.48152	-4.39527	129.70399	1234741.0750
0.1	0.75	55.48152	-4.39527	129.70399	1234740.4884
0.6	0.45	55.48378	-4.39609	129.72800	1235176.7316
0.6	0.55	55.48378	-4.39609	129.72800	1235176.3126
0.6	0.65	55.48378	-4.39609	129.72800	1235175.8098
0.6	0.75	55.48378	-4.39609	129.72800	1235175.2232
1.1	0.45	55.48926	-4.39808	129.78629	1236232.7770
1.1	0.55	55.48926	-4.39808	129.78629	1236232.3580
1.1	0.65	55.48926	-4.39808	129.78629	1236231.8552
1.1	0.75	55.48926	-4.39808	129.78629	1236231.2686

Table 3 presents the impact of reference price effect intensity γ on the retailer's optimal sales price p^*_{ss}, production rate u^*_{ss}, inventory level x_{ss} and profit V_{ss} in the steady state. As shown in Table 3, the optimal sales price p^*_{ss}, inventory level x_{ss} and profit V_{ss} decrease with the reference price intensity, while the production rate u^*_{ss} increases. This implies that under the premise that the stochastic reference price is considered, the large gap between the reference price and the actual sales price will cause the retailer to reduce the sales price, thereby reducing this gap and driving demand. Previous research on deterministic reference price have shown that such a sales strategy of retailers can increase their profit in the short term, whether it is for single-channel retailers or omni-channel retailers (see Li [22]). However, when the uncertainty of consumers' reference price is considered, although the increase of demand reduces the inventory level of retailers and increases production rate, this sales strategy is not conducive to the profitability of retailers in the long run.

Table 3: The effect of reference price effect γ and reference price uncertainty ε on δ, p^*_{ss}, r^*_{ss}, u^*_{ss}, x_{ss} and V_{ss}.

ε	γ	$p^*_{ss}(r^*_{ss})$	u^*_{ss}	x_{ss}	V_{ss}
0.1	1.00	55.71031	-4.85285	133.62728	1244663.4407
0.1	1.25	55.48152	-4.39527	129.70399	1234740.4884
0.1	1.50	55.28763	-4.00748	126.37908	1226350.4577
0.1	1.75	55.12121	-3.67465	123.52535	1219163.6503
0.6	1.00	55.71261	-4.85375	133.65192	1245102.8955
0.6	1.25	55.48378	-4.39609	129.72800	1235175.2232
0.6	1.50	55.28985	-4.00824	126.40255	1226781.2074
0.6	1.75	55.12341	-3.67535	123.54835	1219590.9907
1.1	1.00	55.71818	-4.85591	133.71176	1246170.4090
1.1	1.25	55.48926	-4.39808	129.78629	1236231.2686
1.1	1.50	55.29526	-4.01007	126.45953	1227827.5706
1.1	1.75	55.12875	-3.67705	123.60421	1220629.0705

Table 4 provides the impact of memory factor λ on the retailer's optimal sales price p^*_{ss}, production rate u^*_{ss}, inventory level x_{ss} and profit V_{ss} in the steady state. As shown in Table 3, when the uncertainty of consumers' reference price is considered, the optimal sales price p^*_{ss}, inventory level x_{ss} and profit V_{ss} increase with the memory factor λ, while the production rate u^*_{ss} decreases. The previous studies on deterministic reference price have shown that the retailer should reduce the sales price when consumers are more sensitive to the latest prices, whether it is for single-channel retailers or omni-channel retailers (see Li [22]). However, both Table 3 and Table 4 have shown that the retailer's price reduction strategy is unfavorable to its long-term profitability. From the perspective of long-term sales, the most important thing for retailers is to increase the reference price of consumers in order to reduce the gap between the sales price and the reference price, which is beneficial to the increase of the retailer profit.

Table 5 presents the impact of return rate ξ and ϕ on the retailer's optimal sales price p^*_{ss}, production rate u^*_{ss}, inventory level x_{ss} and profit V_{ss} in the steady state. It follows from Table 5 that the optimal sales price p^*_{ss}, inventory level x_{ss} and profit V_{ss} decrease

Table 4: The effect of memory factor λ and reference price uncertainty ε on p^*_{ss}, r^*_{ss}, u^*_{ss}, x_{ss} and V_{ss}.

ε	λ	$p^*_{ss}(r^*_{ss})$	u^*_{ss}	x_{ss}	V_{ss}
0.1	0.2	55.48152	-4.39527	129.70399	1234740.4884
0.1	0.3	57.49582	-8.42386	164.24501	1322959.1906
0.1	0.4	59.78705	-13.0063	203.53502	1425654.5248
0.1	0.5	62.41656	-18.2653	248.62573	1546590.7015
0.6	0.2	55.48378	-4.39609	129.72800	1235175.2232
0.6	0.3	57.49836	-8.42526	164.27404	1323434.6410
0.6	0.4	59.78993	-13.0084	203.56976	1426178.0272
0.6	0.5	62.41982	-18.2682	248.66702	1547171.6305
1.1	0.2	55.48926	-4.39808	129.78629	1236231.2686
1.1	0.3	57.50456	-8.42867	164.34452	1324589.6094
1.1	0.4	59.79693	-13.0134	203.65412	1427449.7446
1.1	0.5	62.42775	-18.2751	248.76730	1548582.8751

with the return rate ξ and ϕ, while the production rate u^*_{ss} increases. This demonstrates that the increase of return rate makes the retailer have to reduce the sales price, which will increase consumer demand and production rate to some extent. However, in the long run, the increase of return rate will reduce the profit of retailers.

Table 5: The effect of return rate ξ and ϕ on p^*_{ss}, r^*_{ss}, u^*_{ss}, x_{ss} and V_{ss}.

ϕ	ξ	$p^*_{ss}(r^*_{ss})$	u^*_{ss}	x_{ss}	V_{ss}
0.01	0.04	55.49712	-4.42648	129.97151	1235416.3146
0.01	0.05	55.48599	-4.40421	129.78061	1234934.0338
0.01	0.06	55.47486	-4.38195	129.58972	1234451.8353
0.01	0.07	55.46373	-4.35968	129.39884	1233969.7191
0.03	0.04	55.48152	-4.39527	129.70399	1234740.4884
0.03	0.05	55.47039	-4.37301	129.51311	1234258.3230
0.03	0.06	55.45926	-4.35075	129.32223	1233776.2399
0.03	0.07	55.44813	-4.32849	129.13136	1233294.2391
0.05	0.04	55.46592	-4.36408	129.43650	1234064.8240
0.05	0.05	55.45479	-4.34181	129.24562	1233582.7739
0.05	0.06	55.44366	-4.31955	129.05476	1233100.8061
0.05	0.07	55.432531	-4.29729	128.86390	1232618.9207

Table 6 shows the impact of the fraction of online consumers choosing to BOPS σ on the retailer's optimal sales price p^*_{ss}, production rate u^*_{ss}, inventory level x_{ss} and profit V_{ss} in the steady state. It follows from Table 6 that the optimal sales price p^*_{ss}, inventory level x_{ss} and profit V_{ss} increase with σ, while the production rate u^*_{ss} decreases. This implies that if the retailer takes the stochastic reference price into his pricing and production decisions, it is always beneficial to keep a lower production and have a higher price. Otherwise, the model indicates that the retailer would systematically produce more and price too low, and thus lose profit.

Table 6: The effect of the fraction of online consumers choosing to BOPS σ on p^*_{ss}, r^*_{ss}, u^*_{ss}, x_{ss} and V_{ss}.

σ	$p^*_{ss}(r^*_{ss})$	u^*_{ss}	x_{ss}	V_{ss}
0.1	55.47834	-4.38891	129.64946	1234602.71848
0.3	55.48152	-4.39527	129.70399	1234740.48844
0.5	55.48470	-4.40163	129.75854	1234878.26513
0.7	55.48788	-4.40800	129.81308	1235016.04853
0.9	55.49106	-4.41436	129.86762	1235153.83866

Table 7: The effect of shipping fee m and cross-selling profit l on p^*_{ss}, r^*_{ss}, u^*_{ss}, x_{ss} and V_{ss}.

m	l	$p^*_{ss}(r^*_{ss})$	u^*_{ss}	x_{ss}	V_{ss}
2	2	55.48152	-4.39527	129.70399	1234740.4884
2	3	55.48152	-4.39527	129.70399	1234740.4884
2	4	55.48152	-4.39527	129.70399	1234740.4884
2	5	55.48152	-4.39527	129.70399	1234740.4884
3	2	55.48152	-4.39527	129.70399	1234740.4884
3	3	55.48152	-4.39527	129.70399	1234740.4884
3	4	55.48152	-4.39527	129.70399	1234740.4884
3	5	55.48152	-4.39527	129.70399	1234740.4884
4	2	55.48152	-4.39527	129.70399	1234740.4884
4	3	55.48152	-4.39527	129.70399	1234740.4884
4	4	55.48152	-4.39527	129.70399	1234740.4884
4	5	55.48152	-4.39527	129.70399	1234740.48847

Table 7 presents the impact of shipping fee m and cross-selling profit l on the retailer's optimal sales price p^*_{ss}, production rate u^*_{ss}, inventory level x_{ss} and profit V_{ss} in the steady state. It can be seen from Table 7 that the shipping fee m and cross-selling profit l have no effect on the expected steady state optimal sales price p^*_{ss}, production rate u^*_{ss}, inventory level x_{ss} and profit V_{ss}. This is different from the previous study on the deterministic reference price, which showed that the increase in shipping fee m and cross-selling profit l is unfavorable for retailers to increase the sales price. However, when the stochastic reference price is considered, these are not the key factors that retailers consider when making their pricing and production decisions. This provides the necessary guarantee for retailers to implement the strategy accurately.

6. Managerial Insights

In this section, some managerial insights are derived from the previous numerical analysis, which can be adopted by an omni-channel retailer to formulate its pricing and production strategies with stochastic reference price effects.

First, the demand uncertainty δ has a negative impact on the retailer's profit V_{ss} in the steady state. Hence, the retailer can not ignore the impact of demand uncertainty on its retail operations. Especially in the omni-channel era where the mobile Internet is highly developed, the behavior of consumers becomes more uncertain, which will affect the demand forecast. In addition, the demand forecast needs to consider both online and offline consumers. Therefore, it is particularly important for retailers to accurately predict consumer demand in the context of omni-channel era, thereby reducing the impact of demand uncertainty on their profitability.

Second, when the uncertainty of the reference price ε increases, we have known that the uncertainty of the reference price has positive impacts on the retailer's sales price p^*_{ss} and profit V_{ss} in the steady state, the retailer should increase the sales price, although in the short term, it will make consumers feel that the difference between reference price and actual sales price is too large, which will affect their purchase desire, thus leading to an increase in the retailer's inventory level and a decrease in production rate. However, in the long run, with the increase of consumers' reference price, the gap between reference price and actual sales price will decrease, which will eventually benefit the retailer.

Third, if the consumer's ability to remember past price becomes stronger or the gap between the actual sales price and consumers' reference price becomes larger, it illustrates that consumers will become less loyal to the product, which will usually cause the retailer reduce the sales price. Although reducing sales price can stimulate demand, reduce inventory level and increase production rate in the short term. However, just as shown in Table 3 and Table 4, when the uncertainty of consumers' reference price is considered, the markdown policy is not conducive to the retailer's profitability in the long run. It is always beneficial to keep a lower production and have a higher sales price. Otherwise, the retailer would systematically produce more and price too low, and thus lose profit.

Fourth, when the uncertainty of consumers' reference price is considered, the fraction of online consumers choosing to BOPS σ have positive impacts on the optimal sales price p^*_{ss}, inventory level x_{ss} and profit V_{ss} in the steady state. This shows that the retailer can benefit from the advantages of high-quality services and in-store experience provided by offline physical stores. Thus, in addition to laying out online channel for the omni-channel BOPS retailer, it is also necessary to strengthen the service operations of physical stores, such as creating a comfortable physical store experience environment, offering customized service and improving expertise, skill and attitude of the salesmen. Additionally, the increase of return rate makes the retailer have to reduce the sales price, which will increase consumer demand and production rate to some extent. However, in the long run, the increase of return rate will reduce the profit of retailers. Hence, the retailer should provide consumers with high-quality products and services. The measures mentioned above can not only increase the loyalty of consumers to the products, but also help to enhance the credibility of the retailer. Finally, when stochastic reference price is considered, we have known that the shipping fee m and cross-selling profit l are not the key factors that retailers consider when making their pricing and production decisions. This provides the necessary guarantee for retailers to implement precise policies.

7. Conclusion

Our research complements the existing research stream in coordinating pricing and inventory replenishment decisions under omni-channel retail environmental by taking into the consideration of consumers' stochastic reference price effects. Specifically, this paper utilizes the consumers' stochastic reference price in prospect theory to analyze an omni-channel retailer's joint dynamic pricing and production management problem in which consumers can cancel their orders before payment and return the products after payment if the products don't meet their expectation. The omni-channel retailer's optimal pricing and production rate are derived by maximizing the total expected profit under stochastic reference price effects over an infinite horizon in a continuous framework. In addition, a set of sensitivity analysis is discussed to characterize the impacts of system parameters on the optimal decisions and some managerial insights are revealed. Our main results are summarized as follows. First, our analysis shows that the optimal price and production rate are linear feedback form of the two state variables (i.e., inventory level and reference price) when production and inventory/shortage cost are strictly convex. Moreover, the sufficient conditions of stability and monotone convergence properties are derived for the expected steady state inventory and reference price. Second, we investigate how key system parameters affect the optimal decisions. The demand uncertainty has a negative impact on the retailer's profit V_{ss} in the steady state. The uncertainty of the reference price ε has positive impacts on the retailer's sales price p^*_{ss} and profit V_{ss} in the steady state. The reference price effect intensity γ has negative impacts on the optimal sales price p^*_{ss}, inventory level and profit V_{ss}. Furthermore, the memory factor λ and the fraction of online consumers choosing to BOPS σ have positive impacts on the optimal sales price p^*_{ss}, inventory level x_{ss} and profit V_{ss}. The return rate ξ and ϕ have negative impacts on the optimal sales price p^*_{ss}, inventory level x_{ss} and profit V_{ss}. This research fills the gap of behavioral operation in the field of omni-channel joint pricing and inventory management in a continuous framework.

Appendix

The Derivation of the HJB equation (4.1)**.**

Let $V(X,R)$ be the profit-to-go function, when the state variables at time t are (X,R). By the principle of optimality

$$\begin{aligned} V = \max_{u(t),p(t)\in\Omega} \{ & [(\lambda_1((p-s)(1-\xi)-m)+\lambda_2((p-s)(1-\phi)+l)+s)\cdot(\beta_0-\beta_1 p(t)+\gamma(R(t)-p(t))) \\ & -C(u(t))-H(X(t))]dt+e^{-zdt}V(X+dX,R+dR)\}. \end{aligned} \tag{A.1}$$

Using the Taylor series expansion, the term $V(X+dX,R+dR)$ in (A.1) can be written as:

$$\begin{aligned} V(X+dX,R+dR) = & V(X,R)+V_X dX+V_R dR+\frac{1}{2}V_{XX}(dX)^2+\frac{1}{2}V_{RR}(dR)^2 \\ & +\frac{1}{2}V_{XR}dXdR+\text{highter-orderterms}. \end{aligned} \tag{A.2}$$

For convenience, introduce notations $f = u(t) - (\beta_0 - \beta_1 p(t) + \gamma(R(t) - p(t)))$ and $g = \lambda(p(t) - R(t))$. It follows from the multiplication rules of stochastic calculus, we get $(dt)^2 = 0$, $dW(t)dt = 0$, $d(W(t))^2 = dt$, $dW(t)dY(t) = 0$ and $e^{-zdt} = 1 - zdt$, Then

$$(dX)^2 = f^2(dt)^2 + \delta^2(dW(t))^2 + 2f\delta W(t)dt = \delta^2 dt, \tag{A.3}$$

$$(dR)^2 = g^2(dt)^2 + \varepsilon^2(dY(t))^2 + 2g\varepsilon\sqrt{R(t)}dY(t) = R\varepsilon^2 dt, \tag{A.4}$$

$$dXdR = fg(dt)^2 + f\varepsilon\sqrt{R(t)}dY(t)dt + g\delta dW(t)dt + \delta\varepsilon dW(t)dY(t) = 0. \tag{A.5}$$

Substituting (A.2)-(A.5) into (A.1) gives

$$\begin{aligned} V = \max_{u(t),p(t)\in\Omega} \{ & [(\lambda_1((p-s)(1-\xi)-m)+\lambda_2((p-s)(1-\phi)+l)+s)\cdot(\beta_0-\beta_1 p(t)+\gamma(R(t)-p(t))) \\ & -C(u(t))-H(X(t))]dt+(1-zdt)V+V_X dX+V_R dR+\frac{1}{2}\delta^2 V_{XX}dt \\ & +\frac{1}{2}\varepsilon^2 RV_{RR}dt+o(dt)\}. \end{aligned} \tag{A.6}$$

Canceling the term V on both sides of (A.6), dividing the remainder by dt, and letting $dt \to 0$, we obtain the HJB equation (4.1). This completes the proof. □

Proof of Proposition 1. The optimal sales price p^* and production rate u^* can be derived by maximizing the right hand side of HJB equation (4.1) with respective to p and u. □

Proof of Corollary 1. Since $C(u) = c_1 u + c_2 u^2$ and $H(X) = x_1 X + x_2 X^2$ the HJB equation (4.1) becomes

$$\begin{aligned} V = \max_{u(t),p(t)\in\Omega} \{ & [\lambda_1((p-s)(1-\xi)-m)+\lambda_2((p-s)(1-\phi)+l)+s]\cdot[\beta_0-\beta_1 p(t)+\gamma(R(t)-p(t))] \\ & -(c_1 u + c_2 u^2) - (x_1 X + x_2 X^2)+V_x[u(t) - (\beta_0-\beta_1 p(t)+\gamma(R(t)-p(t)))] \\ & +V_R\lambda(p(t) - R(t))+\frac{1}{2}\delta^2 V_{XX}+\frac{1}{2}\varepsilon^2 RV_{RR}\}. \end{aligned} \tag{A.7}$$

Hence, maximizing the right hand side of HJB equation (A.7) with respective to p and u, which gives the result. This completes the proof. □

Proof of Proposition 2.

(i) The Derivation of the Riccati System. When the profit-to-go function $V(X,R)$ is quadratic, i.e., $V(X,R) = a_1 + a_2 X + a_3 X^2 + a_4 XR + a_5 R + a_6 R^2$ by substituting this quadratic expression as well as (4.5) and (4.6) in Corollary 1 into the HJB equation (4.1), and comparing the coefficients of the corresponding terms on both sides of the equation, we can get the following Riccati System of a_i $(i = 1, \dots, n)$:

$$\begin{aligned} 4\Delta_1^2 c_2 z a_1 = {} & 4\Delta_1 c_2[\alpha_1\Delta_2+(\beta_1+\gamma)\alpha_1 a_2]+4\Delta_1 c_2\lambda\alpha_1 a_5-4c_2(\beta_1+\gamma)(1-\lambda x_1-\lambda_2\phi)\alpha_1^2 \\ & +4\Delta_1^2 c_2[(\lambda_1\xi+\lambda_2\phi)s\beta_0+(\lambda_1 m+\lambda_2 l)\beta_0-\beta_0 a_2+\delta^2 a_3]-2\Delta_1^2 c_1(a_2-c_1) \\ & -\Delta_1^2(a_2-c_1)^2+2\Delta_1^2(a_2-c_1)a_2, \\ 2\Delta_1^2 c_2 z a_2 = {} & 2\Delta_1 c_2[\alpha_2\Delta_2+(\beta_1+\gamma)(a_2+2\alpha_1 a_3)+\lambda_1(\alpha_1 a_4+\alpha_2 a_5)-4\alpha_1\alpha_2 c_2(\beta_1+\gamma) \\ & \times(1-\lambda_1\xi-\lambda_2\phi)-2\Delta_1^2 a_2 a_3-2\Delta_1^2 c_2 x_1+\Delta_1^2(4a_2 a_3-2c_1 a_3)-4\Delta_1^2 c_2\beta_0 a_3, \\ \Delta_1^2 c_2 z a_3 = {} & \Delta_1 c_2[2\alpha_2(\beta_1+\gamma)a_3+\lambda_1\alpha_2 a_4]-c_2\alpha_2^2(\beta_1+\gamma)(1-\lambda_1\xi-\lambda_2\phi)-\Delta_1^2 c_2 x_2+\Delta_1^2 a_3^2, \\ \Delta_1^2 c_2 z a_4 = {} & \Delta_1 c_2[(\beta_1+\gamma)(2\alpha_3 a_3+\alpha_2 a_4)+\lambda_1(\alpha_3 a_4+2\alpha_2 a_6)+\Delta_1^2 a_3 a_4-\Delta_1^2 c_2(2\gamma a_3+\lambda a_4) \\ & -2c_2\alpha_2\alpha_3(\beta_1+\gamma)(1-\lambda_1\xi-\lambda_2\phi), \end{aligned}$$

$$\begin{aligned}4\Delta_1^2 c_2 z a_5 =& 4\Delta_1 c_2[\alpha_3\Delta_2+(\beta_1+\gamma)(\alpha_3 a_2+\alpha_1 a_4)+\lambda_1(\alpha_3 a_5+2\alpha_1 a_6)+(1-\lambda_1\xi-\lambda_2\phi)\gamma\alpha_1]\\ &-8c_1\alpha_1\alpha_3(\beta_1+\gamma)(1-\lambda_1\xi-\lambda_2\phi)+4\Delta_1^2 c_2[(\lambda_1\xi+\lambda_2\phi)s\gamma+\beta_0(\lambda_2 l-\lambda_1 m)\gamma]\\ &-2\Delta_1^2\alpha_2 a_4+2\Delta_1^2(2a_2a_4-c_1a_4+a_4^2)-4\Delta_1^2 c_2(a_2+\beta_0 a_4)+4\Delta_1^2 c_2(\lambda a_5-\varepsilon^2 a_6),\\ \Delta_1^2 c_2 z a_6 =& \Delta_1 c_2\alpha_3[(\beta_1+\gamma)a_4+2\lambda_1 a_6+(1-\lambda_1\xi-\lambda_2\phi)\gamma]-c_2\alpha_3^2(\beta_1+\gamma)(1-\lambda_1\xi-\lambda_2\phi)\\ &-\Delta_1^2 a_4^2-\Delta_1^2 c_2(\gamma a_4+2\lambda a_6),\end{aligned}$$

where $\Delta_1 = (\beta_1+\gamma)(1-\lambda_1\xi-\lambda_2\phi) > 0$, $\alpha_4 = (1-\lambda_1\xi-\lambda_2\phi)\beta_0 + (\beta_1+\gamma)a_2 + \lambda a_5 - (\lambda_1\xi + \lambda_2\phi)s$, $a_2 = 2(\beta_1+\gamma)a_3 + \lambda a_4$, $\alpha_3 = (1-\lambda_1\xi-\lambda_2\phi)\gamma + (\beta_1+\gamma)a_4 + 2\lambda a_6$.

(ii) Since $V_X = a_2 + 2a_3X + a_4R$ and $V_R = a_4X + a_5 + 2a_6R$, substituting V_X and V_R into (4.5) in Corollary 1 yields (4.8).

(iii) Substituting $V_X = a_2 + 2a_3X + a_4R$ into (4.6) in Corollary 1 yields (4.9). This completes the proof. □

Proof of Proposition 3. Substituting the optimal strategies (4.8) and (4.9) into (3.5) and (3.1) yields:

$$\begin{aligned}dX =& \Big[\Big(\frac{a_2-c_1}{2c_1}-\beta_0+\frac{\beta_1+\gamma}{\Delta_1}\alpha_1\Big)+\Big(\frac{a_3}{c_2}+\frac{\beta_1+\gamma}{\Delta_1}\alpha_2\Big)X+\Big(\frac{a_4}{2c_2}+\frac{\beta_1+\gamma}{\Delta_1}\alpha_3-\gamma\Big)R\Big]dt\\ &+\delta dW(t), \qquad (A.8)\\ dR =& \frac{\lambda\alpha_1}{\Delta_1}+\frac{\lambda\alpha_2}{\Delta_1}X+\Big(\frac{\lambda\alpha_3}{\Delta_1}-\lambda\Big)R+\varepsilon\sqrt{R(t)}dY(t). \qquad (A.9)\end{aligned}$$

Taking expectation of both sides of (A.8) and (A.9), we have

$$\begin{aligned}\dot{X} =& \frac{a_2-c_1}{2c_1}-\beta_0+\frac{\beta_1+\gamma}{\Delta_1}\alpha_1+\Big(\frac{a_3}{c_2}+\frac{\beta_1+\gamma}{\Delta_1}\alpha_2\Big)x+\Big(\frac{a_4}{2c_2}+\frac{\beta_1+\gamma}{\Delta_1}\alpha_3-\gamma\Big)r, \qquad (A.10)\\ \dot{r} =& \frac{\lambda\alpha_1}{\Delta_1}+\frac{\lambda\alpha_2}{\Delta_1}x+\Big(\frac{\lambda\alpha_3}{\Delta_1}-\lambda\Big)r, \qquad (A.11)\end{aligned}$$

where $x = E(X)$ and $r = E(R)$. Setting $\dot{x} = 0$ and $\dot{r} = 0$ in (A.10) and (A.11) and solving the resulting two equations in two unknowns x and r gives the expression of r_{ss}^* and x_{ss}. Moreover, taking expectation of both sides of (3.1) and setting the resulting equation to zero gives $p_{ss}^* = r_{ss}^*$. This completes the proof. □

Proof of Proposition 4. The Jacobian matrix for the linear system of equations $\dot{x} = 0$ and $\dot{r} = 0$ in (A.12) and (A.13) is as follows:

$$J = \begin{bmatrix} \frac{a_3}{c_2}+\frac{\beta_1+\gamma}{\Delta_1}\alpha_2 & \frac{a_34}{2c_2}+\frac{\beta_1+\gamma}{\Delta_1}\alpha_3-\gamma \\ \frac{\lambda\alpha_2}{\Delta_1} & \frac{\lambda\alpha_3}{\Delta_1}-\lambda \end{bmatrix}$$

Thus, the determinant and trace of Jacobian matrix J are

$$\det(J) = \frac{2\lambda(\alpha_3-\Delta_1)[\Delta_1 a_3 + c_2\alpha_2(\beta_1+\gamma)] - \lambda\alpha_2[\Delta_1 a_4 + 2c_2(\beta_1+\gamma) - 2\gamma c_2\Delta_1]}{2c_2\Delta_1^2}$$

and

$$\mathrm{tr}(J) = \frac{\Delta_1 a_3 + c_2\alpha_2(\beta_1 + \gamma) + \lambda c_2(\alpha_3 - \Delta_1)}{c_2\Delta_1}$$

Using the stability criteria of Section 3.3 in Strange [33], if $\det(J) > 0$ and $\mathrm{tr}(J) < 0$, then the expected steady state inventory x_{ss} and reference price r^*_{ss} are asymptotically stable. Moreover, if $[\mathrm{tr}(J)]^2 - 4\det(J) \geq 0$, the stable steady state inventory x_{ss} and reference price r^*_{ss} are monotonically convergent. If $[\mathrm{tr}(J)]^2 - 4\det(J) < 0$, the convergence is with transient oscillations. This completes the proof. □

Acknowledgements

The authors thank the editors and two anonymous referees for their valuable comments and suggestions that substantially improved this paper. This study is partially supported by the Science and Technology Commissioner of Fujian Province and their team funding, the Natural Science Foundation of Hebei Province under grant G2019203387 and the Doctoral Research Start-up Foundation of Inner Mongolia University for The Nationalities under grant BS653.

References

[1] Arnold, L. (1974). Stochastic Differential Equation: Theory and Applications, New York, Wellesley, USA.

[2] Bi, W, Li, G. and Liu, M.(2017). *Dynamic pricing with stochastic reference price effects based on a finite memory window*, International Journal of Production Research, Vol.55, 3331-3348.

[3] Cao, Y. and Duan, Y. R. (2020). *Joint production and pricing inventory system under stochastic reference price effect*, Computers & Industrial Engineering, Vol.143, 1-12.

[4] Chen, X., Hu, P., Shum, S. and Zhang, Y. (2016). *Dynamic stochastic inventory management with reference price effects*, Operations Research, Vol.64, 1529-1536.

[5] Chen, X., Hu, Z. Y. and Zhang, Y. H. (2019). *Dynamic pricing with stochastic reference price effect*, Journal of the Operations Research Society of China, Vol.7, 107-125.

[6] Choi, *t*. M. and Guo, S. (2017). *Responsive supply in fashion mass customisation systems with consumer returns*, International Journal of Production Research, Vol.56, 3409-3422.

[7] Fan, C., Liu, Y. M., Yang, X. H., Chen, X. H. and Hu, J. H. (2019). *Online and offline cooperation under buy-online, pick-up-in-store: Pricing and inventory decisions*, Journal of Industrial and Management Optimization, Vol.15, 1455-1472.

[8] Fibich, G., Gavious, A. and Lowengart, O. (2003). *Explicit solutions of optimization models and differential games with nonsmooth* (*asymmetric*) *reference-price effects*, Operations Research, Vol.51, 721-734.

[9] Forrester Research. (2014). *Customer desires vs. Retailer capabilities: Minding the omni-channel commerce gap*, Forrester Consulting.

[10] Gimpl-Heersink, L., Rudloff, C., Fleischmann, M. and Taudes, A. (2008). *Integrating pricing and inventory control: Is it worth the effort*? BuR-Business Research, Vol.1, 106-123.

[11] Güler, M. G., Bilgiç, *t*. and Güllü, R. (2015). *Joint pricing and inventory control for additive demand models with reference effects*, Annals of Operations Research, Vol.226, 255-276.

[12] Halzack, S. (2015). *From new mobile offerings to a potential off-price business, a look at what's in store for Macy's*, Washington Post, January 13, available at https://www.washingtonpost.com/news/business/wp/2015/01/13/from-new-mobile-offerings-to-a-potential-off-price-business-a-look-at-whats-in-store-for-macys/.

[13] Harsha, P. and Subramanian, S. (2018). U.S. Patent Application No. 15/199, 205.

[14] Harsha, P., Subramanian, S. and Yichanco, J. (2019). *Dynamic pricing of omnichannel inventories*, Manufacturing & Service Operations Management, Vol.21, 47-65.

[15] He, P., He, Y. and Xu, H. (2020). *Buy-online-and-deliver-from-store strategy for a dual-channel supply chain considering retailer's location advantage*, Transportation Research Part E: Logistics and Transportation Review, Vol.144, 1-21.

[16] He, Y., Huang, H. F. and Li, D. (2020). *Inventory and pricing decisions for a dual-channel supply chain deteriorating products*, Operational Research, Vol.20, 1461-1503.

[17] He, Y., Xu, Q. Y. and Wu, P. K. (2020). *Omnichannel retail operations with refurbished consumer returns*, International Journal of Production Research, Vol.58, 271-290.

[18] Herbon, A. and Kogan, K. (2014). *Time-dependent and independent control rules for coordinated production and pricing under demand uncertainty and finite planning horizons*, Annals of Operations Research, Vol.223, 195-216.

[19] Jin, M., Li, G. and Cheng, *t*. C. E. (2018). *Buy online and pick up in-store: Design of the service area*, European Journal of Operations Research, Vol.268, 613-623.

[20] Kalyanaram, G. and Winer, R. S. (1995). *Empirical generalizations from reference price research*, Marketing Science, Vol.14, 161-169.

[21] Li, R. H. and Teng, J. *t*. (2018). *Pricing and lot-sizing decisions for perishable goods when demand depends on selling price, reference price, product freshness, and displayed stocks*, European Journal of Operational Research, Vol.270, 1099-1108.

[22] Li, Y. (2021). *Multiperiod dynamic pricing and inventory control decisions for an omnichannel BOPS retailer with reference price effects*, Mathematical Problems in Engineering, Vol.2019, 1-21.

[23] Li, Z. H., Wang, W. S., Jin, H. S. and Wang, D. (2021). *Omnichannel retailing operations with coupon promotions*, Journal of Retailing and Consumer Service, Vol.58, 1-13.

[24] Liu, L. R. and Xu, Q. (2020). *Joint decisions on pricing and ordering for omnichannel BOPS retailers: Considering online returns*, Sustainability, Vol.12, 1-18.

[25] Pei, Z., Toombs, L. and Yan, R. (2014). *How does the added new online channel impact the supporting advertising expenditure*, Journal of Retailing and Consumer Services, Vol.21, 229-238.

[26] Peng, H. F. and Huang, M. Y. (2014). *Retail business model innovation and its path analysis under the new technology environment-Taking Suning cloud business as an example* (in Chinese), Macroeconomic Research, Vol.2, 108-115.

[27] Ren, H. and Huang, T. (2018). *Modeling customer bounded rationality in operations management: A review and research opportunities*, Computers & Operations Research, Vol.91, 48-58.

[28] Rigby, D. (2011). *The future of shopping*, Harvard Business Review, Vol.89, 64-75.

[29] Rosenblum, P. and Kilcourse, B. (2013). Omni-channel 2013: The long road to adoption, RSR 2013 Benchmark Report, Retail System Research.

[30] Saghiri, S., Wilding, R., Mena, C. and Bourlakis, M. (2017). *Toward a three-dimensional framework for omni-channel*, Journal of Business Research, Vol.77, 53-67.

[31] Song, H. F., Ran, L. and Shang, J. (2017). *Multi-period optimization with loss-averse customer behavior: Joint pricing and inventory decisions with stochastic demand*, Expert Systems with Applications, Vol.72, 421-429.

[32] Statista. (2019). B2C e-commerce sales worldwide from 2012-2018, http://www.statista.com/statistics/261245/b2c-e-commerce-sales-worldwide/.

[33] Strang, G. (2014). Differential equations and linear algebra, Wellesley-Cambridge, Wellesley, MA, USA.

[34] UPS. (2015). UPS online shopping study: Empowered consumers changing the future of retail, Press release, June 3, United Parcel Service of America, Atlanta.

[35] Wallace, D. W., Giese, J. L. and Johnson, J. L. (2014). *Customer retailer loyalty in the context of multiple channel strategies*, Journal of Retailing, Vol.80, 249-263.

[36] Wang, Q., Zhao, N. G., Wu, J. and Zhu, Q. Y. (2018). *Optimal pricing and inventory policies with reference price effect and loss-averse customers*, Omega: The International Journal of Management Science, Vol.99, 1-16.

[37] Wu, L. and Wu, D. (2016). *Dynamic pricing and risk analytics under competition and stochastic reference price effects*, IEEE Transactions on Industrial Informatics, Vol.12, 1282-1293.

[38] Wu, S., Liu, Q. and Zhang, R. Q. (2015). *The reference effects on a retailer's dynamic pricing and inventory strategies with strategic consumers*, Operations Research, Vol.63, 1320-1335.

[39] Zhang, J., Chiang, W. K. and Liang, L. (2014). *Strategic pricing with reference effects in a competitive supply chain*, Omega: The International Journal of Management Science, Vol.44, 126-135.

[40] Zhang, L. Z., Xu, Q. Y. and He, Y. (2018). *Omnichannel retail operations with consumer returns and order cancellation*, Transportation Research E: Logistics and Transportation Review, Vol.118, 308-324.

[41] Zhang, P., He, Y. and Zhao, X. (2019). *Preorder-online, pickup-in-store" strategy for a dual-channel retailer*, Transportation Research Part E: Logistics and Transportation Review, Vol.122, 27-47.

[42] Zhang, S. C., Zhang, J. X., Shen, J. and Tang, W. S. (2019). *A joint dynamic pricing and production model with asymmetric reference price effect*, Journal of Industrial and Management Optimization, Vol.15, 667-688.

College of Management, Yang-en University, Quanzhou, Fujian province, PRC.

E-mail: 30457978@qq.com

Major area (s): Marketing management and consumer behavior.

College of Mathematics and Physics, Inner Mongolia University for The Nationalities, Tongliao, The inner mongolia autonomous region, PRC.

E-mail: imunliyuan@163.com

Major area (s): Supply chain and logistics management, behavior operation management, variational inequalities and complementarity problems.

College of Management, Yang-en University, Quanzhou, Fujian province, PRC.

E-mail: hym@ysu.edu.cn (Corresponding author)

Major area(s): Supply chain and logistics management, medical, health and pension management.

(Received January 2022; accepted June 2022)