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# Optimal Dynamic Pricing Strategy for Inventory with Reference Price Effects

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**Keywords**

Dynamic pricing
Inventory
Reference price effects
Pontryagin's maximum principle

**Abstract.**

A dynamic pricing model of a retailer selling a kind of fixed quantity of items with shelf life under reference price effects is investigated. An optimal control model is established to maximize the retailer's total profit, where the demand is time-varying and depends on price and reference price. The continuous time dynamic optimal pricing policy with reference price effects are obtained for both finite and infinite planning horizon by Pontryagin's maximum principle. Numerical experiments are account for the impacts of the key system parameters. The theoretical and numerical analysis shows that: (1) in finite planning horizon, the optimal price is increasing in the initial reference price and reference price coefficient, and is concave in memory effect coefficient and discount rate, while the optimal inventory level decreases with these parameters. (2) in infinite planning horizon, the retailer makes a choice between price skimming and price penetration based on the difference between the consumers' initial reference price and the steady state price. Further, the steady state price increases with memory effect coefficient and discount rate, and decreases with reference price coefficient. Finally, some managerial inspiration and implication that retailer may adopt to formulate its pricing policy are obtained.

## 1. Introduction

Integrating decisions about pricing and inventory can significantly improve the profits of firms. Hence, it always been a major strategy of many firms such as Dell, Amazon, FairMarket, etc. (see Feng [14]). Additionally, the studies on behavioral sciences (see Kalyanaram and Winer [21]) have recognized that consumers are subject to anchoring effects. They indicated that consumers will remember the past prices of one product when repeated transactions and develop price expectations of the product which, captured by the reference price, is a benchmark when consumers purchase the product. If the current sales price is lower (higher) than the reference price, consumers are more likely (less inclined) to make the purchase. This phenomenon is usually called the reference price effects. Consumers are called loss averse (loss neutral) if demand is more responsive to consumers' perceived losses than (as) their perceived gains. Otherwise, they are called loss seeking. In practice, the reference price effect has an important impact on demand and thus becomes an indispensable part of firms' decision making (see Mathies and Gudergan [25]). Thus, it is interesting and necessary to investigate the joint pricing and inventory decisions with the consideration of reference price effects.

Because of the significant effects of the reference price on consumers' purchasing behaviors, reference price effects have received a great deal of attention from practitioners and researchers. There exist a large number of literatures that study the pricing decisions with reference price effects. This line of research started in the 1990s. Greenleaf [16] first analyzed the firm's pricing strategy with reference price effects and explained how the reference price affects the promotion decision of a firm during a sales period. Some recent works explore how pricing strategies should account for the reference price effects. For more details refer to Bi et al. [2]; Chen et al. [5, 7]; Hu et al. [20]; Zha et al. [35]; Zhang et al. [37]; Zhao et al. [38] and the references therein. However, to our best knowledge, only a few papers have integrated reference price effects into the pricing and inventory control model, such as Gler et al. [17, 18]; Chen et al. [6]; Xue et al. [32]; Chenavaz and Paraschiv [12]. Most of them focus on periodic-review inventory systems, the research on continuous time pricing and inventory system with reference price effects is still very limited, which motivate us to do the explore in this aspect.

In this paper, we study a dynamic pricing model of a retailer selling a fixed amount of inventory with reference price effects. The purpose of this paper is to find the optimal dynamic pricing strategies under reference price effects so that the total profit is maximized. The continuous time dynamic optimal pricing strategy with reference price effects are obtained for finite planning horizon by applying Pontryagin's maximum principle. Moreover, as an extension, the infinite planning horizon pricingmodel is investigated and the characteristics of the optimal price in the steady state is analyzed. Finally, numerical experiments are employed to illustrate the impact of consumers' initial reference price, reference price parameters (the memory effect coefficient and the reference price coefficient) and discount rate on the optimal pricing strategies.

The differences between our study and existing literature considering continuous time joint pricing and inventory with reference price effects (including Xue et al. [32] and Chenavaz and Paraschiv [12]) are mainly reflected in the following three aspects. First, although Xue et al. [32] considered inventory, but they studied the pricing decision of perishable goods. We study the pricing strategies of products with shelf life, such as milk formula, tea and snacks, etc. The shelf life of such goods is usually longer than that of perishable goods, such as a few months, one year or more. Second, we analyze the impact of consumers' initial reference price $r_0$, reference price parameters (the memory effect coefficient and the reference price coefficient) and discount rate on the optimal pricing strategies and the optimal inventory level. These parameters have important impacts on the formulation of retailers' pricing and inventory strategies, which are not reflected in Chenavaz and Paraschiv [12]. Third, we characterize the steady state solution of the optimal price when the planning period is infinite, which are not discussed in the above two papers.

The rest of the paper is organized as follows. Section 2 reviews related literature. Problem description and assumptions are given in Section 3. The finite and infinite planning horizon optimal control models for dynamic pricing with reference price effects are developed in Section 4 and Section 5, respectively. Numerical examples are given in Section 6, and some managerial insights are presented in Section 7. Section 8 concludes this paper and possible extension of future research. The proof of all conclusions are presented in Appendix.

## 2. Literature Review

This work is mainly related to two streams of literature: one delves into joint pricing and inventory models while the other discusses the reference price effects in the field of joint pricing and inventory control models.

The first stream of research focuses on the joint pricing and inventory policy. This stream of research is mainly divided into two aspects: periodic review and continuous review. The research on periodic review inventory strategy can be traced back to the 1950s. Whitin [31] first combined dynamic pricing with inventory control strategy, and analyzed the economic order quantity model under price-dependent demand. Recent literature on joint pricing and inventory strategy can be summarized as follows: Chen et al. [11] studied a class of joint pricing and inventory strategies with costly price adjustments under stochastic demand. Zhu [39] investigated the inventory strategy of returns and expediting, and the results showed that the optimal inventory policy is a modified base-stock policy. Chen et al. [8] studied the joint pricing and inventory strategy of perishable goods, they derived the boundaries of the optimal order-up-to level, and provided an effective heuristic strategy. Chung [13] considered the dynamic pricing and stochastic inventory model of multi-period discounted commodities. Bernstein et al. [1] studied a near-optimal heuristic algorithms for joint pricing and stochastic inventory with lead time. Shen et al. [27] considered a joint pricing and inventory control problem based on a general random price-dependent demand. Hu et al. [19] proposed a joint pricing and inventory control problem, where production incurs a fixed cost plus a convex or concave variable cost. For continuous review inventory system, Chen and Simchi-Levi [9] established a semi-Markov inventory model, in which demand was a random variable with independent time intervals in discrete time. Chao and Zhou [4] analyzed a infinite-horizon continuous review stochastic inventory model in which the demand process is Poisson with a price-dependent arrival rate. The closed form solutions of optimal inventory and optimal pricing strategies were obtained. Yin and Rajaram [34] considered the emergency ordering model and proved the optimality of $(s, S, p)$ strategy. Mamani and Moizadeh [24] found that the expediting replenishment could improve system efficiency due to the reduction of conventional replenishment. Yao [33] used the upper bound method to prove the optimality of $(s, S, p)$ strategy for Brownian motion joint pricing and inventory problems. Cao and Yao[3] considered the joint drift rate control and impulse control for a stochastic inventory system under long-run average cost criterion. Liu et al. [23] developed a revenue and cost-sharing contract to coordinate supply chain pricing and inventory under consumer balking and price-dependent to achieve the Pareto-improvement. A more detailed review of this line of research is provided in recent papers by Chen and Simchi-Levi [10] and Simchi-Levi [28].

The second stream of research considers the reference price effects in the field of joint pricing and inventory systems. This line of research started with Gimpl-Heersink [15], who proved the optimality of the base-stock-list-price for the single-period and two-period models. However, the optimality of the base-stock-list-price was stricter for the multi-period setting. Urban [30] analyzed a single-period joint pricing and inventory model with symmetric and asymmetric reference price effects and showed that the consideration of the reference price had a substantial impact on the firm's profitability. Even if the single-period profit function is non-concave, Zhang [36] proved the optimality of the base-stock-list-price policy by a class of transformation technique. Taudes and Rudloff [29] provided an application of the two-period model from Gimpl-Heersink [15] to electronic commodities. Gler et al. [17] extended the model of Gimpl-Heersink [15] to the concave demand function, where the concavity of the revenue function could be maintained via a transformation technique. The optimality of the state-dependent order-up-to strategy was proved for the transformed concave revenue function model. Gler et al. [18] used the safety stock as a decision variable to characterize the steady state solution to the problem when the planning horizon is infinite. Chen et al. [6] introduced a new concave transform technique to ensure the concavity of the profit function by using the preservation property of supermodularity in parameter optimization problems with the nonlattice structure, and then proved the optimality of the base-stock-list-price strategy. Li and Hou [22] considers a pricing and inventory problem with regular and expedited supplies under reference price effects. The above pricing and inventory models with reference price effects mainly focus on periodic-review. However, continuous time pricing and inventory system considering reference price effects is more recent. Xue et al. [32] studied the problem of dynamic pricing with inventory for deteriorating items, where the demand function is assumed to be linear. Chenavaz and Paraschiv [12] considered a dynamic pricing problem with a fixed inventory without replenishment, where the demand function is assumed to be more general. For other related works in this stream of research, interested readers may refer to the review by Ren and Huang [26].

From the literature review, although either the dynamic pricing and inventory strategies or reference price effects in the field of joint pricing and inventory systems are well developed by these papers, few of them delve into the discussion of continuous time joint pricing and inventory decision with reference price effects. It is worthwhile to further study the reference price effects (including the memory effect coefficient and the reference price coefficient) on pricing and inventory strategies, especially in the steady state. This gives us reason to investigate the continuous time joint strategy of both the pricing and inventory with the effects of reference price.

## 3. Problem Description and Assumption

Consider a retailer selling a kind of product with shelf life to the end consumers. The retailer, who face the consumers' reference price effects, controls the retail price $p(t)$ for a fixed amount of inventory without possibility of replenishment over an finite planning horizon.

Let $r(t)$ denote the consumer's reference price, according to Zhang et al. [37] and Xue et al. [32], the reference price is modeled as continuous weighted average of past prices with an exponentially decaying weighting function, namely

$$r(t) = \beta e^{-\beta t} \int_{-\infty}^{t} e^{\beta s} p(s) ds,$$

where $\beta > 0$ is called "memory effect" parameter ($0 \leq \beta \leq 1$), which represents the degree of forgetfulness about the product's prior prices. A higher $\beta$ indicates that consumers are more impacted by the product's latest prices, i.e., the consumers are lower loyalty. If $\beta = 0$, consumers never remember any past prices and the reference price $r(t)$ will remain at the initial reference price $r(0) = r_0$, which is a constant. An immediate consequence of the equation above is that reference price formation is given by the following ordinary differential equation

$$\dot{r}(t) = \beta\big(p(t) - r(t)\big). \tag{3.1}$$

As in Zhang et al. [37] and Xue et al. [32], the demand function in the presence of reference price effects is given by

$$Q(t) = a - \delta p(t) - \gamma\big(p(t) - r(t)\big), \tag{3.2}$$

where $a > 0$ and $\delta > \gamma > 0$. The parameter $a$ denotes the basic market size, $\delta$ represents the effect intensity of the current sales price, and $\gamma$ reflects the reference price effects. A higher $\gamma$ implies that consumers are more sensitive to the gap between the sales price and reference price $\big(p(t) - r(t)\big)$, $\gamma = 0$ represents having no reference price effects.

Let $I(t)$ represent the retailer's inventory level at time $t$. Because the product with shelf life will also deteriorate, but the rate of deterioration is slower than that of the perishable product. Therefore, the natural deteriorating coefficient $\theta > 0$ is assumed to be small. Thus, the serviceable inventory loss at rate $f\big(I(t)\big) = \theta I(t)$ and the inventory level variances can be expressed as

$$\dot{I}(t) = -Q(t) - f\big(I(t)\big), \quad I(0) = I_0, \tag{3.3}$$

where $I(0) = I_0 > 0$ is the given initial inventory level.

The inventory holding cost $C(I)$ is a linear function of the current inventory level as

$$C(I) = hI,$$

where $h > 0$ represents the holding cost of items per unit.

The parameters and variables used in this paper are summarized in Table 1; other notations will be defined as needed.

Table 1: Summary of notations.

| Notation | Description |
|---|---|
| *Decision variables* | |
| $p(t)$ | The retailer's retail price at time $t$. |
| $I(t)$ | The retailer's inventory level at time $t$. |
| *Parameters* | |
| $Q(t)$ | The demand at time $t$. |
| $a$ | The basic market size. |
| $\delta$ | The effect intensity of the current retail price. |
| $r(t)$ | The consumer's reference price at time $t$. |
| $\beta$ | The memory effect parameter ($\beta > 0$). |
| $\gamma$ | The reference price effects parameter($\gamma > 0$). |
| $\rho$ | The discount rate. |
| $h$ | The holding cost of items per unit. |

In the following sections, we present the optimal joint pricing and inventory strategies for finite and infinite planning horizons, respectively.

## 4. Optimal Strategies for Finite Planning Horizon

In this section, we address the retailer's optimization problem within the finite planning horizon $[0, T]$. The retailer's objective is to make the pricing strategy for the fixed amount of inventory by maximizing the present value of its profit over the finite planning horizon $[0, T]$, specified as

$$\begin{aligned} \max_{p(\cdot),I(\cdot)} \quad & J = \int_0^T e^{-\rho t}\big[\big(p(t)-c\big)Q(t) - C(I)\big]dt \\ & \quad = \int_0^T e^{-\rho t}\big[\big(p(t)-c\big)\big(a-\delta p(t)-\gamma\big(p(t)-r(t)\big)\big) - hI(t)\big]dt, \\ \text{s.t.} \quad & \dot{r} = \beta\big(p(t)-r(t)\big),\ r(0)=r_0, \\ & \dot{I}(t) = -Q(t) - \theta I(t),\ I(0)=I_0, \\ & I(t) \geq 0. \end{aligned} \tag{4.1}$$

where $\rho > 0$ stands for the discount rate, which is an exogenous constant and determined by the cost of capital. $c$ represents for the unit cost of the product.

The optimal pricing policy with reference price effects can be characterized by solving the optimization problem (4.1), which is shown in the following proposition.

**Proposition 1.** *The optimal dynamic pricing strategy $p_c^*$, the optimal inventory level $I_c^*$ and corresponding reference price path $r_c^*$ during the planning horizon $[0, T]$ are given by*

$$p^*(t) = \frac{1}{\Delta_1}\big[(\rho+\beta)(a+\delta c) + \rho c\gamma\big] + \frac{\gamma}{2(\delta+\gamma)}(c_1^* e^{mt} + c_2^* e^{\tilde{m}t})$$

$$+\frac{h}{(\rho+\theta)\Delta_1}(\rho\delta+\beta\delta+\rho\gamma)\big[e^{(\rho+\theta)(t-T)}-1\big], \tag{4.2}$$

$$I^*(t)=c_3^*e^{-\theta t}-\frac{a}{\theta}+\frac{\delta}{\theta\Delta_1}\big[(\rho+\beta)(a+\delta c)+\rho c\gamma\big]+\frac{1-(2\delta+\gamma)}{2(\delta+\gamma)}\Big[\frac{1}{m}(c_1^*e^{(m-\theta)t}+\frac{1}{\tilde{m}}c_2^*e^{(\tilde{m}-\theta)t}\Big]$$
$$+\frac{h}{(\rho+\theta)\Delta_1}(\rho\delta+\beta\delta+\rho\gamma)\Big[\frac{1}{\rho+2\theta}e^{(\rho+\theta)(t-T)}-\frac{1}{\theta}\Big], \tag{4.3}$$

*and*

$$r^*(t)=\frac{1}{\Delta_1}[(\rho+\beta)(a+\delta c)+\rho c\gamma]+c_1^*e^{mt}+c_2^*e^{\tilde{m}t}+\frac{h}{(\rho+\theta)\Delta_1}(\rho\delta+\beta\delta+\rho\gamma)\big[e^{(\rho+\theta)(t-T)}-1\big], \tag{4.4}$$

*where*

$$\Delta_1=2\delta(\rho+\beta)+\rho\gamma,$$
$$c_1^*=\frac{Q_2}{Q_1-Q_2}(\Delta_2-r_0),\qquad c_2^*=\frac{Q_1}{Q_1-Q_2}(r_0-\Delta_2),$$
$$Q_1=\Big[\frac{2\delta+\gamma}{\beta}+\frac{2(\delta+\gamma)}{\beta^2}m\Big]e^{mT},\qquad Q_2=\Big[\frac{2\delta+\gamma}{\beta}+\frac{2(\delta+\gamma)}{\beta^2}\tilde{m}\Big]e^{\tilde{m}T},$$
$$\Delta_2=\frac{1}{\Delta_1}\Big[(\rho+\beta)(a+\delta c)+\rho c\gamma+\frac{h}{\rho+\theta}(\beta\delta+\rho\delta+\rho\gamma)\big[e^{-(\rho+\theta)T}-1\big]\Big],$$
$$m=\frac{\rho}{2}-\sqrt{\Big(\frac{\rho}{2}+\beta\Big)\Big(\frac{\rho}{2}+\eta\beta\Big)},\quad \tilde{m}=\frac{\rho}{2}+\sqrt{\Big(\frac{\rho}{2}+\beta\Big)\Big(\frac{\rho}{2}+\eta\beta\Big)},\quad \eta=\frac{\delta}{\delta+\gamma},$$
$$c_3^*=I_0+\frac{a}{\theta}-\frac{\delta}{\theta\Delta_1}[(\rho+\beta)(a+\delta c)+\rho c\gamma]-\frac{1-(2\delta+\gamma)}{2(\delta+\gamma)}\Big(\frac{1}{m}c_1^*+\frac{1}{\tilde{m}}c_2^*\Big)$$
$$-\frac{h}{(\rho+\theta)\Delta_1}(\rho\delta+\beta\delta+\rho\gamma)\Big[\frac{1}{\rho+2\theta}e^{-(\rho+\theta)T}-\frac{1}{\theta}\Big].$$

From (4.2), (4.3) and (5.4), we find that it is difficult to deduce the relationship between the results and the variable directly, so we will give a detailed analysis in the numerical section, i.e., Section 6, including the impact of initial reference price $r_0$, reference price parameters (the memory effect $\beta$ and the reference price effect $\gamma$), and discount rate $\rho$ on the optimal pricing and optimal inventory level.

## 5. Optimal Strategies for Infinite Planning Horizon

In this section, we extend the above model to the infinite planning horizon. The optimization problem for the retailer can be expressed as

$$\max_{p(\cdot),I(\cdot)} J=\int_0^{+\infty}e^{-\rho t}\big[\big(p(t)-c\big)Q(t)-C(I)\big]dt$$
$$=\int_0^{+\infty}e^{-\rho t}\big[\big(p(t)-c\big)\big(a-\delta p(t)-\gamma\big(p(t)-r(t)\big)\big)-hI(t)\big]dt, \tag{5.1}$$
$$\text{s.t.}\quad \dot{r}=\beta\big(p(t)-r(t)\big),\ r(0)=r_0,$$
$$\dot{I}(t)=-Q(t)-\theta I(t),\ I(0)=I_0,\ I(t)\geq 0.$$

where $\rho > 0$ stands for the discount rate, $c$ represents for the unit cost of the product.

With a similar method presented in Section 4, we obtain the optimal pricing policy with reference price effects by solving the optimization problem (5.1), which is shown in the following proposition.

**Proposition 2.** *The optimal dynamic pricing strategy $p_c^*$, the optimal inventory level $I_c^*$ and corresponding reference price path $r_c^*$ during the planning horizon are given by*

$$p_c^* = \overline{p}_c^{ss} + (r_0 - \overline{p}_c^{ss})\Big(1 + \frac{m}{\beta}\Big)e^{mt}, \tag{5.2}$$

$$I_c^* = c_4^* e^{-\theta t} + \frac{1}{m+\theta}\Big[(\delta+\gamma)\Big(1 + \frac{m}{\beta}\Big) - \gamma\Big](r_0 - \overline{p}_c^{ss})e^{mt} - \frac{1}{\theta}(a - \delta\overline{p}_c^{ss}), \tag{5.3}$$

*and*

$$r_c^* = \overline{p}_c^{ss} + (r_0 - \overline{p}_c^{ss})e^{mt}, \tag{5.4}$$

*where*

$$\begin{aligned}
\overline{p}_c^{ss} &= \frac{1}{\Delta_1}\Big[(\rho+\beta)(a + c\delta) + \rho c\gamma - \frac{h(\beta\delta + \rho\delta + \beta\gamma)}{\rho+\theta}\Big],\\
m &= \frac{\rho}{2} - \sqrt{\Big(\frac{\rho}{2} + \beta\Big)\Big(\frac{\rho}{2} + \eta\beta\Big)}, \qquad \eta = \frac{\delta}{\delta+\gamma},\\
\Delta_1 &= 2\delta(\rho+\beta) + \rho\gamma = (\rho+\beta)\delta + (\delta+\gamma)(\rho+\eta\beta),\\
c_4^* &= I_0 + \frac{1}{\theta}(a - \delta\overline{p}_c^{ss}) - \frac{1}{m+\theta}\Big[(\delta+\gamma)\Big(1+\frac{m}{\beta}\Big) - \gamma\Big](r_0 - \overline{p}_c^{ss}).
\end{aligned}$$

From Proposition 2, we can obtain the following insights.

(1) The retailer makes a choice between two pricing strategies (price skimming and price penetration) based on the difference between the consumers' initial reference price $r_0$ and $\overline{p}_c^{ss}$. When $r_0 > \overline{p}_c^{ss}$, the retailer adopt price skimming strategy. Otherwise, the retailer adopt price penetration strategy (see the proof of Proposition 2). This reflects the importance of consumers' initial reference price $r_0$ to the retailer's pricing strategy formulation.

(2) In the steady state, i.e., $t \to \infty$, the consumers' reference price is the same as current sales price. From (5.2), we can derive that $\overline{p}_c^{ss}$ is the steady state price. Under the steady state, $\overline{p}_c^{ss}$ should be equal to the optimal price when there is no reference price, i.e., $p_{no-rp}^* = (a + \delta c)/2\delta$. However, by simple comparison, we find that $\overline{p}_c^{ss}$ is no higher than $p_{no-rp}^* = (a + \delta c)/2\delta$ and $\overline{p}_c^{ss} = p_{no-rp}^*$ is available only when both the discount rate $\rho$ and the holding cost per unit $h$ are equal to zero. This is to say, either $\rho > 0$ or $h > 0$ will lead to $\overline{p}_c^{ss} < p_{no-rp}^*$. The intuition is that, on one side, when current income is higher than future profits, the gains owing to demand increasing are higher than the discounted losses caused by lowering the price in subsequent periods, which induces the retailer to reduce the steady state price. On the other side, the high profit brought by the increase in demand compared with the high inventory holding cost, firms are more inclined to reduce inventory holding cost by lowing the price in the steady state.

(3) The steady state price $\overline{p}_c^{ss}$ is lower than the stable state price $p_c^{ss}$ obtained by Zhang et al. [37], where the inventory factor is ignored. This indicates that considering inventory and related cost will make the retailer lower the steady state price.

(4) From (5.3), the steady state inventory level is $\bar{I}_c^{ss} = -(a - \delta\overline{p}_c^{ss})/\theta$, which may be negative. This indicates that the inventory replenishment is necessary when the planning horizon is infinite. However, to our best knowledge, there is no research on joint pricing and inventory strategy with reference price effects in the existing literature in infinite planning horizon. Thus, how to replenish inventory in the infinite planning horizon is still a problem worthy of study. A heuristic and feasible strategy may be the $(s, S)$ strategy proposed by Yao [33]. Although we have not established an effective inventory replenishment strategy for infinite planning horizon, our analysis can still reveal the pricing strategy when considering inventory. This is because the optimal price path (5.2) is only related to the inventory holding cost $h$, but not to the quantity of inventory replenishment.

The impact of initial reference price $r_0$, discount rate and reference price parameters (the memory effect $\beta$ and the reference price coefficient $\gamma$) on the optimal pricing strategy is similar to that of Section 4, which we omit here. Based on the above analysis, we are more concerned about the impact of these parameters on the steady state optimal price $\overline{p}_c^{ss}$. For the expression of $\overline{p}_c^{ss}$, we have the following results.

**Proposition 3.** *The steady state price $\overline{p}_c^{ss}$ is not related to the initial reference price $r_0$, but $\overline{p}_c^{ss}$ decreases with the reference price coefficient $\gamma$.*

Proposition 3 indicates that the steady state price $\overline{p}_c^{ss}$ is lower when consumers are more sensitive to the price gap $\big(p(t)-r(t)\big)$. The intuition is as follows. When consumers are sensitive to the difference between price and reference price, the retailer will reduce the price in order to obtain a positive reference price effect, which makes the steady state price lower.

The impact of memory effect $\beta$ and discount rate $\rho$ on steady state price $\overline{p}_c^{ss}$ cannot be obtained directly, so we will examine it through Example 2 in the next section.

## 6. Numerical Experiments

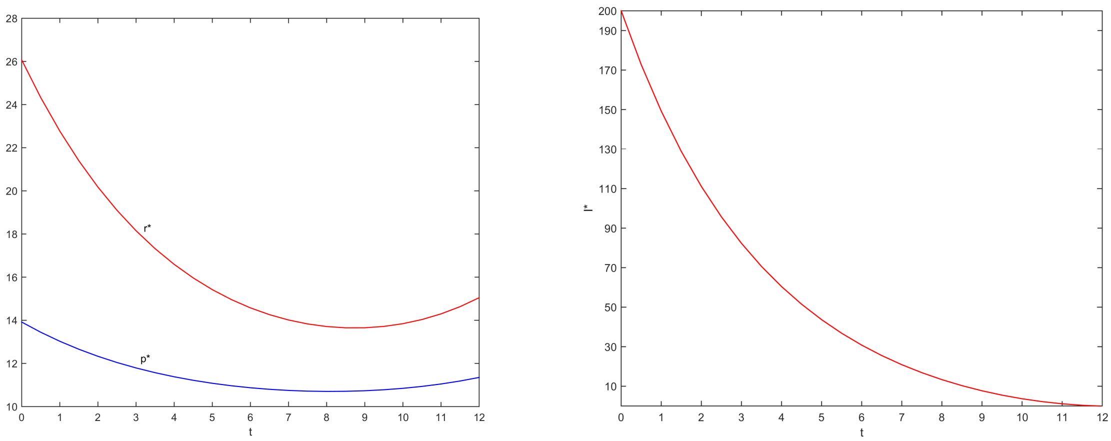
In this section, we present several numerical experiments to illustrate the impact of initial reference price $r_0$, reference price parameters (the memory effect $\beta$ and the reference price effect $\gamma$), and discount rate $\rho$ on the optimal strategies. All experiments below are performed in MATLAB R2014b on a laptop with an Intel(R) Core (TM) i5-7200U central processing unit CPU (2.50 GHz, 2.70GHz) and 8.0 GB of RAM running 64-bit Windows 10 Enterprise.

**Example 1.** Consider a system with the stationary initial parameter values: $a = 20$, $\delta = 1.5$, $\beta = 0.25$, $\gamma = 0.3$, $\rho = 0.15$, $\theta = 0.1$, $r_0 = 26$, $I_0 = 200$, $c = 5$, $h = 1.5$, $T = 12$. The optimal price $p^*$, optimal reference price $r^*$ and optimal inventory level $I^*$ are

$$p^*(t) = 4.4784e^{-0.228t} + 4.644e^{0.25t-4} + 9.354,$$

$$r^*(t) = 16.485e^{-0.228t} + 4.644e^{0.25t-3} + 9.354,$$
$$I^*(t) = 93.437e^{-0.1t} + 150.2923e^{-0.3285t} + 13.2686e^{0.25t-3} - 44.39.$$

With the expressions above, we can get the curves of their changing characteristics, which are shown in Figure 1. It can be seen from Figure 1 that the optimal price $p^*$ and optimal reference price $r^*$ have the same changing trend, and the optimal price path is below the optimal reference price path, which makes the retailer's optimal pricing strategy always maintain a positive reference price effects. In addition, at the beginning of a planning horizon, the gap between the retailer's sales price and the consumers' reference price is large, so the inventory decreases quickly. As the reference price decreases, the gap between the sales price and the reference price becomes smaller. Since then, the reference price increases slowly as the sales price increases, which makes their difference still small. Hence, the inventory level decreases slowly.

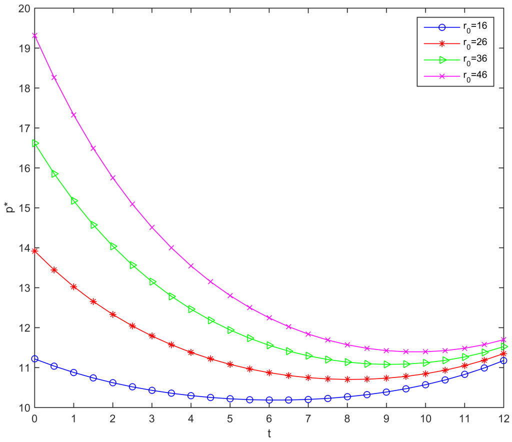



(a) The path of optimal price $p^*$ and reference price $r^*$. (b) The path of optimal inventory level $I^*$.

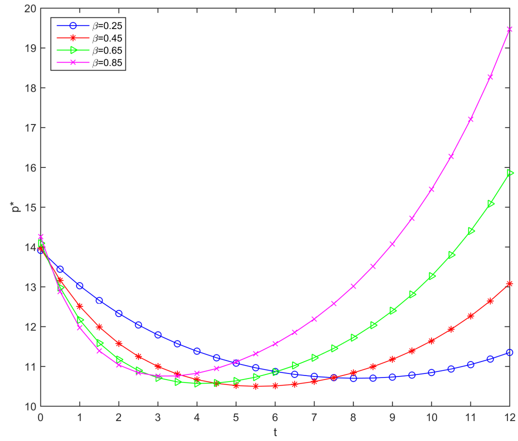
Figure 1: The optimal price $p^*$, reference price $r^*$ and inventory level $I^*$.

Figure 2 presents the effects of $r_0$, $\beta$, $\gamma$, $\rho$ on optimal price $p^*$, where these four parameters are taken separately from sets $\{16, 26, 36, 46\}$, $\{0.25, 0.45, 0.65, 0.85\}$, $\{0.3, 0.6, 0.9, 1.2\}$ and $\{0.05, 0.35, 0.65, 0.95\}$ while keeping the other parameters fixed at the initial parameter values. From Figure 2, we can observe the following managerial insights.

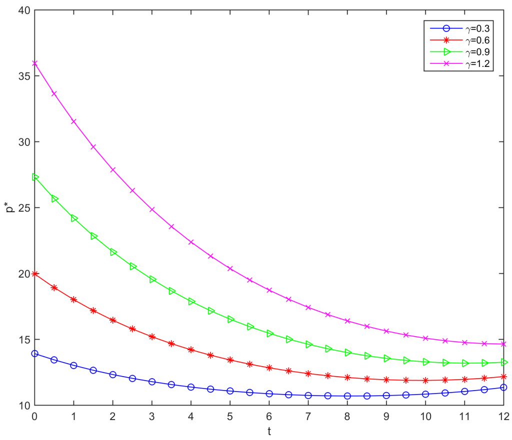
(1) Figure 2(a) reveals the impact of initial reference price $r_0$ on the optimal price $p^*$. It is shown from Figure 2(a) that the optimal price increases with $r_0$. Figure 2(a) also indicates that the optimal price decreases more and more quickly as $r_0$ increases at the beginning of the sales cycle. This can be explained by the fact that when the consumers' initial reference price $r_0$ is high, the retailer adopts a strategy of selling at a low price, which makes the difference between the sales price and the reference price very large, thereby stimulating demand and increasing profit.



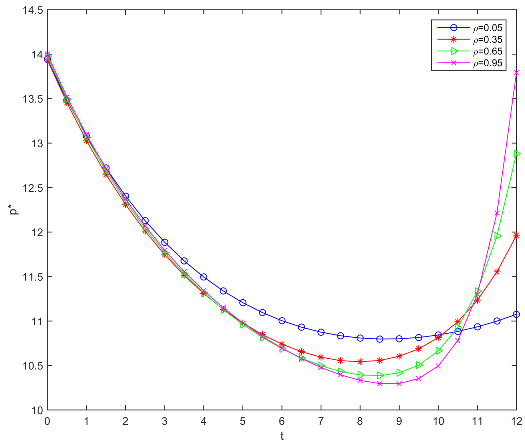

(a) Impact of initial reference price $r_0$ on optimal price $p^*$.




(b) Impact of memory effect $\beta$ on optimal price $p^*$.




(c) Impact of reference price effect $\gamma$ on optimal price $p^*$.




(d) Impact of discount rate $\rho$ on optimal price $p^*$.

Figure 2: The impact of $r_0$, $\beta$, $\gamma$, $\rho$ on optimal price $p^*$.

(2) Figure 2(b) presents the impact of memory effect $\beta$ on the optimal price $p^*$. From Figure 2(b), we can see that the optimal price decreases with $\beta$ at the beginning of sales cycle, as time goes on, the optimal price increases with $\beta$. This indicates that when $\beta$ is large, consumers are more impacted by the product's latest prices, i.e., the consumers are lower loyalty, the retailer should lower the sales price, and gradually increase the sales price as consumers become more loyal.

(3) Figure 2(c) provides the impact of reference price coefficient $\gamma$ on the optimal price $p^*$. Figure 2(c) shows that the optimal price increases as $\gamma$ increases. However, we find that the bigger $\gamma$, the faster the optimal price reduces. This can be interpreted as follows. When $\gamma$ is large, which means the effect of the gap between sales price and

reference price is great, the retailer should lower the sales price below the reference price to increase demand and make more profit.

(4) Figure 2(d) reveals the impact of discount rate $\rho$ on the optimal price $p^*$. The changing trend of $p^*$ with $\rho$ is the same as that of $p^*$ with $\beta$. The intuition is that, when current income is higher than future profits, the gains owing to demand increasing are higher than the discounted losses caused by lowering the price in subsequent periods, which induces the retailer to reduce the sales price.

Figure 3 presents the effects of $r_0$, $\beta$, $\gamma$, $\rho$ on optimal inventory level $I^*$, where these four parameters are taken separately from sets $\{10, 16, 26, 36\}$, $\{0.15, 0.2, 0.25, 0.3\}$, $\{0.3, 0.6, 0.9, 1.2\}$ and $\{0.05, 0.1, 0.15, 0.2\}$ while keeping the other parameters fixed at the initial parameter values. From Figure 3, we can observe the following managerial insights.

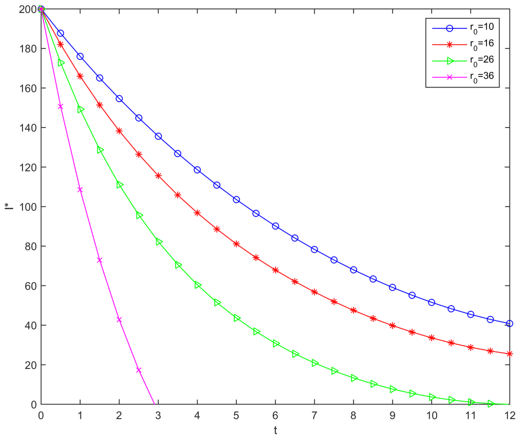
(1) Figure 3(a) reveals the impact of initial reference price $r_0$ on the optimal inventory level $I^*$. It is shown from Figure 3(a) that the optimal inventory level $I^*$ decreases with $r_0$. The optimal sales termination time is also shortened as $r_0$ increases. This result can be directly followed from Figure 2(a) that when the consumers' initial reference price $r_0$ is high, the retailer's low sales price strategy, which makes the difference between the sales price and the reference price very large, stimulates the demand. This leads to the rapid reduction of inventory level, and the optimal sales termination time will also be shortened.

(2) Figure 3(b) presents the impact of memory effect $\beta$ on the optimal inventory level $I^*$. From Figure 3(b), we can see that the optimal inventory level increases with $\beta$. The optimal sales termination time also becomes longer as $\beta$ increases. This demonstrates that when $\beta$ is large, i.e., consumers are less loyal to the product, the optimal inventory level drops slowly and the optimal sales termination time will be longer.

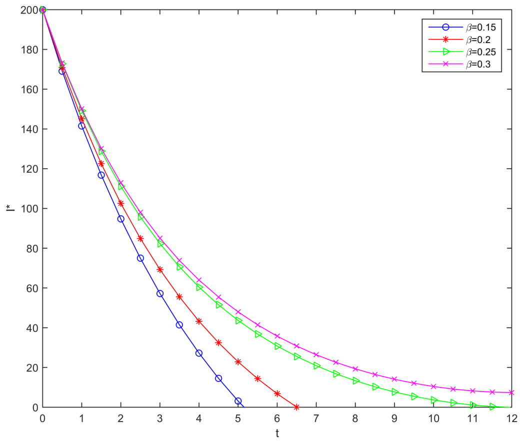
(3) Figure 3(c) provides the impact of reference price coefficient $\gamma$ on the optimal inventory level $I^*$. Figure 3(c) shows that the optimal inventory level decreases with $\gamma$. However, the optimal sales termination time becomes shorter as $\gamma$ increases. This result can be directly followed from Figure 2(c) that the retailer's low sales price strategy stimulates the demand, which leads to the rapid reduction of inventory level, and the optimal sales termination time will also be shortened.

(4) Figure 3(d) reveals the impact of discount rate $\rho$ on the optimal inventory level $I^*$. The changing trend of $I^*$ with $\rho$ is the same as that of $I^*$ with $\beta$. This illustrates that if the retailer is more patient, she is more likely to make the sales price higher because $\partial p^*/\partial \rho < 0$ (Figure 2(d)). Hence, the inventory level is decreases slowly, and the optimal sales termination time will also be longer.

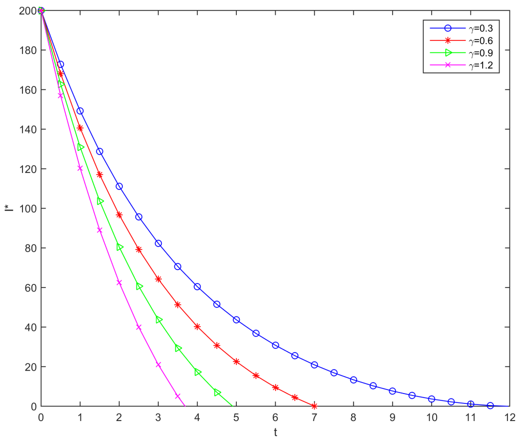
Figure 4 presents the effects of $r_0$, $\beta$, $\gamma$ on optimal profit $J^*$. From Figure 4(a) and Figure 4(c), we can observe that the optimal profit $J^*$ increases with $r_0$ and $\gamma$. This intuition can be found from the interpretation of Figure 2(a) and Figure 2(c). From Figure 4(b), we can observe that the optimal profit $J^*$ decreases with $\beta$. This indicates that consumers' disloyalty to products will force retailers to lower their sales profit, which leads to a lower revenue. Consequently, $r_0$ and $\gamma$ has positive impacts on profit, while $\beta$ has a negative impact on profit.



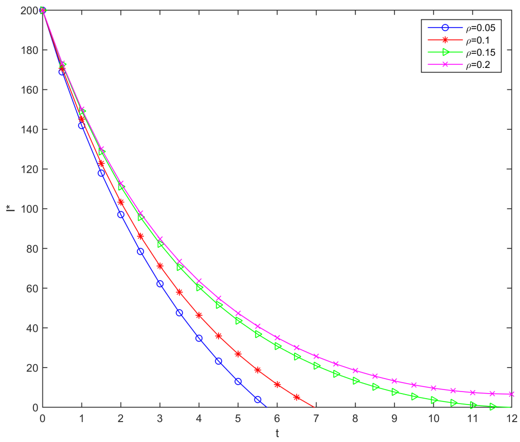

(a) Impact of initial reference price $r_0$ on optimal price $I^*$.




(b) Impact of $\beta$ on optimal inventory level $I^*$.




(c) Impact of reference price effect $\gamma$ on optimal inventory level $I^*$.




(d) Impact of discount rate $\rho$ on optimal inventory level $I^*$.

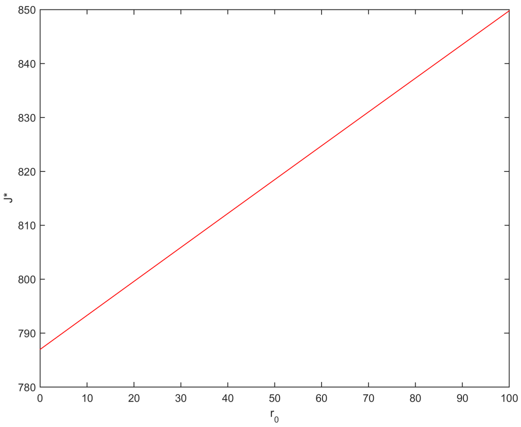
Figure 3: The impact of $r_0$, $\beta$, $\gamma$, $\rho$ on optimal inventory level $I^*$.

**Example 2.** Consider a system with the following initial parameter values: $a = 20$, $\delta = 0.25$, $\beta = 0.25$, $\gamma = 1.5$, $\rho = 0.15$, $\theta = 0.1$, $c = 5$, $h = 1.5$, $T = \infty$. The optimal steady state price can be calculated as $\overline{p}_c^{ss} = 38.375$. Moreover, $r_0$ is chosen in the set $\{26, 56\}$. If $r_0 = 26 < \overline{p}_c^{ss} = 38.375$ ($r_0 = 56 > \overline{p}_c^{ss} = 38.375$), the retailer adopt price penetration (price skimming) strategy. When $r_0 = 26$, the optimal price and reference price are

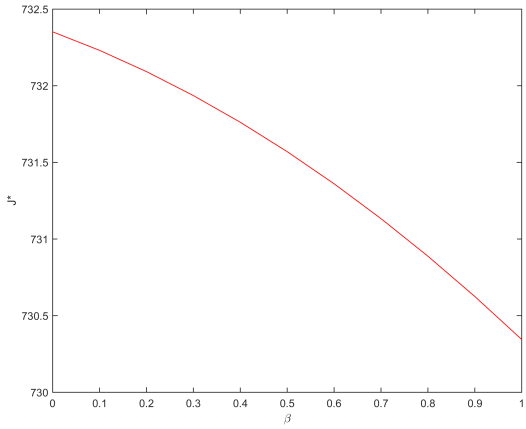
$$p_c^* = -6.6979e^{-0.1147} + 38.375, \qquad r_c^* = -12.375e^{-0.1147} + 38.375.$$

When $r_0 = 56$, the optimal price and reference price are

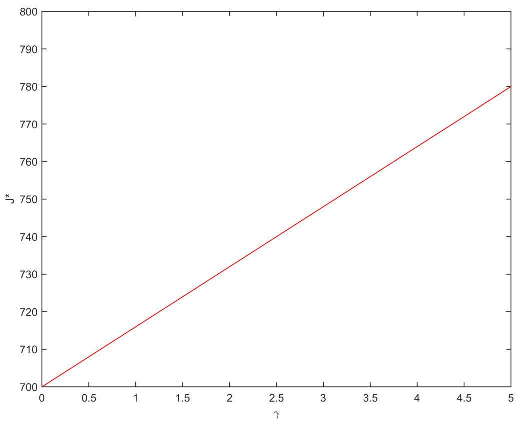
$$p_c^* = 95394e^{-0.1147} + 38.375, \qquad r_c^* = 17.625e^{-0.1147} + 38.375.$$




(a) Impact of initial reference price $r_0$ on optimal profit $J^*$.




(b) Impact of memory effect $\beta$ on optimal profit $J^*$.

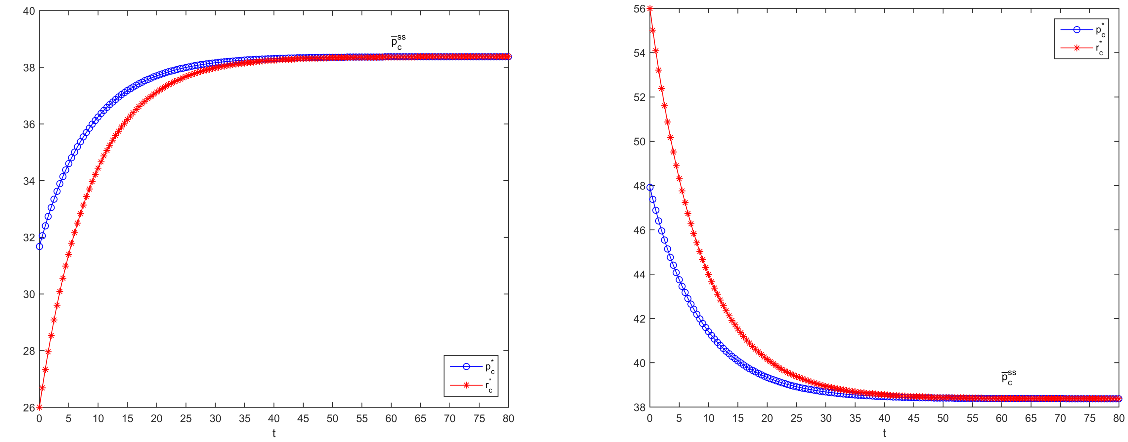



(c) Impact of reference price effect $\gamma$ on optimal profit $J^*$.

Figure 4: The impact of $r_0$, $\beta$, $\gamma$ on optimal profit $J^*$.

Based on the expressions above, we can draw the optimal price and optimal reference price trajectories, as shown in Figure 5. Figure 5 shows that when the retailer make long-term strategies, their pricing strategy is either price penetration or price skimming strategy, which is different from that of finite planning horizon. The main reason for adopting this strategy is to keep consumers' reference price stable so as to avoid making them feel a sense of loss.

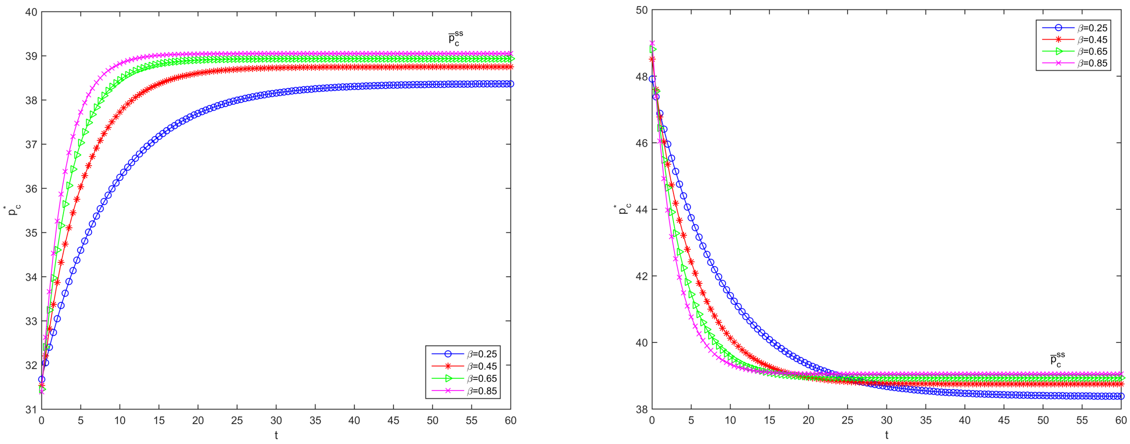
In order to obtain the impact of memory effect $\beta$ and discount rate $\rho$ on steady state price $\overline{p}_c^{ss}$. We vary $\beta$ in the set $\{0.25, 0.45, 0.65, 0.85\}$ or $\rho$ in $\{0.05, 0.35, 0.65, 0.95\}$ while keeping the other parameters fixed at the initial values. The impact of $\beta$ and $\rho$ on $\overline{p}_c^{ss}$ are shown in Figure 6 and Figure 7, respectively. From Figure 6 and Figure 7, we can see that the steady state price $\overline{p}_c^{ss}$ is increasing in both $\beta$ and $\rho$. Therefore, we conclude that $\overline{p}_c^{ss}$ is higher when consumers are less loyal or the retailer is more patient.




(a) $r_0 = 26$, $\overline{p}_c^{ss} = 38.375$, price penetration strategy.

(b) $r_0 = 56$, $\overline{p}_c^{ss} = 38.375$, price skimming strategy.

Figure 5: The optimal price $p_c^*$ and reference price $r_c^*$ path in infinite planning horizon.



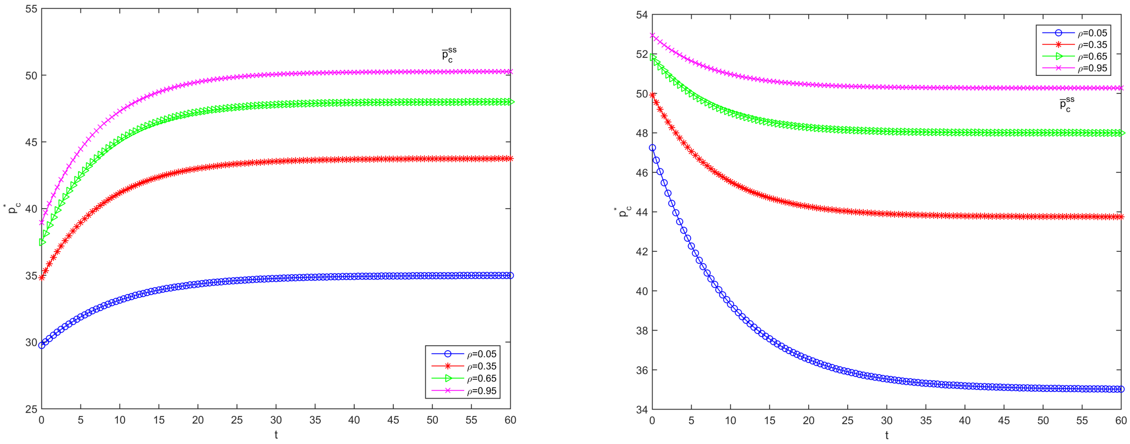

(a) $r_0 = 26$, $\overline{p}_c^{ss} = 38.375$, price penetration strategy.

(b) $r_0 = 56$, $\overline{p}_c^{ss} = 38.375$, price skimming strategy.

Figure 6: The impact of memory effect $\beta$ on the steady state price $\overline{p}_c^{ss}$.

Figure 6 can be explained as follows. (1) When consumers' initial reference price $r_0$ is very low (Figure 6(a)), they tend to be less loyal. Retailers often do not lower the sales price, which further reduces consumers' valuation of the product. Instead, they will gradually increase the sales price in order to increase consumers' valuation, which makes the steady state price higher. (2) When consumers' initial reference price $r_0$ is very high (Figure 6(b)), the more disloyal consumers, the more retailers will lower the sales price so as to obtain a positive reference price effect. But they don't pull the sales price much lower to avoid a very low valuation of consumers, i.e., to avoid them becoming disloyal again.

Figure 7 can be explained as follows. (1) When consumers' initial reference price $r_0$ is very low (Figure 7(a)), as explained in Figure 6(a), retailers will gradually increase the sales price in order to increase consumers' valuation. Moreover, a more patient retailer likely to keep the sales price at a relatively high level, which makes the steady state price higher. (2) When consumers' initial reference price $r_0$ is very high (Figure 7(b)), as explained in Figure 6(b), they will lower the sales price so as to obtain a positive reference price effect, but patient retailers don't pull the sales price much lower to avoid a very low valuation of consumers, which makes the steady state price relatively higher.




(a) $r_0 = 26$, $\overline{p}_c^{ss} = 38.375$, price penetration strategy.

(b) $r_0 = 56$, $\overline{p}_c^{ss} = 38.375$, price skimming strategy.

Figure 7: The impact of discount rate $\rho$ on the steady state price $\overline{p}_c^{ss}$.

## 7. Management Inspiration and Implication

In this section, we provide some management inspiration derived from previous analysis, which can be adopted by retailers when making their pricing strategies with reference price effects.

(1) Initial reference price $r_0$ has a positive impact on the optimal price and profit. When the consumers' initial reference price $r_0$ is higher, retailers can increase the sales price of product to make profits.

(2) Memory effect parameters $\beta$ have a negative impact on profit. When consumers are not loyal to the product, retailers should first adopt the strategy of low price (price skimming strategy) to establish consumers' loyalty to the product, and then gradually increase the price (price penetration strategy).

(3) The reference price effect coefficient $\gamma$ has a positive impact on profit. When the difference between the sales price and the reference price is large, which means that the effect of the gap between sales price and reference price is great, retailers should

lower the sales price below the reference price to increase demand and make more profit (price skimming strategy).

(4.1) The discount rate $\rho$ have a negative impact on profit. When current income is higher than future profits, the gains owing to demand increasing are higher than the discounted losses caused by lowering the price in subsequent periods, retailers should reduce the sales price (price skimming strategy) first to increase demand, and then gradually increase the sales price when demand stabilizes (price penetration strategy).

## 8. Conclusions

This paper studies a dynamic pricing problem of a retailer selling a fixed amount of inventory with reference price effects. The continuous time dynamic optimal pricing strategy with reference price effects are derived by applying Pontryagin's maximum principle. Furthermore, numerical experiments are employed to illustrate the impact of consumers' initial reference price $r_0$, reference price parameters (the memory effect $\beta$ and the reference price coefficient $\gamma$) and discount rate $\rho$ on the optimal pricing strategy, the optimal inventory level and the retailer's profit. Finally, we extend the finite planning horizon model to the infinite planning horizon and analyze the impacts of these parameters on optimal steady state price.

The results of this study are summarized as follows. First, in finite planning horizon, the optimal price $p^*$ is increasing in the initial reference price $r_0$ and reference price coefficient $\gamma$, and is concave in memory effect coefficient $\beta$ and discount rate $\rho$, while the optimal inventory level decreases with these parameters. Second, in infinite planning horizon, the retailer makes a choice between price skimming and price penetration based on the difference between the consumers' initial reference price $r_0$ and the steady state price $\overline{p}_c^{ss}$. When $r_0 > \overline{p}_c^{ss}$, the retailer adopt price skimming strategy. Otherwise, the retailer adopt price penetration strategy. Further, the steady state price $\overline{p}_c^{ss}$ increases with memory effect coefficient $\beta$ and discount rate $\rho$, and decreases with reference price coefficient $\gamma$. Third, the optimal profit in finite planning horizon $J^*$ increases as $r_0$ and $\gamma$ increase, while $J^*$ decreases $\beta$ as increases.

Though this paper has identified the effects of reference price on a retailer's pricing strategies when selling a kind of fixed quantity of items with shelf life, there are still some shortcomings that can be investigated in the future. First, how to replenish inventory in the infinite planning horizon is not solved in this paper, so the continuous time pricing and inventory with the influence of reference price effects in the infinite planning horizon is still a problem worthy of study. Second, this study regards the parameter "$h$: The holding cost of items per unit" as a constant quantity. But in reality, "$h$" should be a variable that can reflect the increase or decrease of market supply and demand. From (4.2), we can see that the optimal price $p^*$ is decreasing in "$h$". However, whether $p^*$ is still a decreasing function of "$h$" when "$h$" changes remains to be further studied. Hence, an interesting future research topic is to examine the dynamic effect of inventory holding cost "$h$" on retailer's pricing strategies, which will increase considerable research value in this field. Third, in our study, the customers' reference price can be observed by retailers. However, the information on customers' reference price is difficult to get in reality. Thus, demand learning can be incorporated into formulating pricing strategies in the presence of the reference price effects. Finally, with the rapid development of information technology centered on the mobile Internet, consumers' purchase patterns are also diversified. In this case, how to study the reference price of consumers on firms' pricing and inventory decisions is also one of the interesting and meaningful research directions in the future.

**Acknowledgements**

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## Appendix

**Proof of Proposition 1.** The current-value Hamiltonian function for the retailer over the sales period $[0,T]$ is given by

$$\begin{aligned} H(I,r,p\lambda_1,\lambda_2) &= \big[(p-c)\big(a-\delta p-\gamma(p-r)\big)-hI\big]+\lambda_1\beta(p-r)+\lambda_2\big(-Q(t)-\theta I\big) \\ &= \big[(p-c)\big(a-\delta p-\gamma(p-r)\big)-hI\big]+\lambda_1\beta(p-r) \\ &\quad+\lambda_2\big(-a+\delta p+\gamma(p-r)-\theta I\big), \end{aligned} \tag{A.1}$$

where $\lambda_1$ and $\lambda_2$ are the adjoint variables associated with the state variables $r$ and $I$, respectively.

Applying general control theory, we have the following optimization conditions:

$$\frac{\partial H}{\partial p}=0 \Leftrightarrow p=\frac{c+\lambda_2}{2}+\frac{a+\gamma r+\lambda_1\beta}{2(\delta+\gamma)}, \tag{A.2}$$

$$\dot{\lambda}_1=\rho\lambda_1-\frac{\partial H}{\partial r}=(\rho+\beta)\lambda_1-\gamma(p-c-\lambda_2), \tag{A.3}$$

$$\dot{\lambda}_2=\rho\lambda_2-\frac{\partial H}{\partial r}=(\rho+\theta)\lambda_2+h. \tag{A.4}$$

Substituting (A.2) into (A.3) and (4.1) , we get

$$\dot{\lambda}_1=\Big[\rho+\frac{2\delta+\gamma}{2(\delta+\gamma)}\beta\Big]\lambda_1+\frac{\gamma}{2}\lambda_2-\Big[\frac{\alpha}{2(\delta+\gamma)}-\frac{c}{2}\Big]\gamma-\frac{\gamma^2}{2(\delta+\gamma)}r, \tag{A.5}$$

$$\dot{r}=\frac{\beta^2}{2(\delta+\gamma)}\lambda_1+\frac{\beta}{2}\lambda_2-\frac{2\delta+\gamma}{2(\delta+\gamma)}\beta r+\Big[\frac{\alpha}{2(\delta+\gamma)}+\frac{c}{2}\Big]\beta. \tag{A.6}$$

Solving the first-order linear differential equation (A.4) with the boundary condition $\lambda_2(T)=0$, we can obtain

$$\lambda_2=\frac{h}{\rho+\theta}\big[e^{(\rho+\theta)(t-T)}-1\big], \tag{A.7}$$

substituting (A.7) into (A.5) and (A.6), respectively, gives

$$\dot{\lambda}_1=\Big[\rho+\frac{2\delta+\gamma}{2(\delta+\gamma)}\beta\Big]\lambda_1-\frac{\gamma^2}{2(\delta+\gamma)}r+\frac{\gamma h}{2(\rho+\theta)}\big[e^{(\rho+\theta)(t-T)}-1\big]-\Big[\frac{a}{2(\delta+\gamma)}-\frac{c}{2}\Big]\gamma,$$

$$\dot{r}=\frac{\beta^2}{2(\delta+\gamma)}\lambda_1-\frac{2\delta+\gamma}{2(\delta+\gamma)}\beta r+\frac{\beta h}{2(\rho+\theta)}\big[e^{(\rho+\theta)(t-T)}-1\big]+\Big[\frac{a}{2(\delta+\gamma)}+\frac{c}{2}\Big]\beta.$$

Thus, we derive $\begin{bmatrix}\dot{\lambda}_1\\ \dot{r}\end{bmatrix}=A\begin{bmatrix}\lambda_1\\ r\end{bmatrix}+b$, where $A=\begin{bmatrix}\rho+\dfrac{2\delta+\gamma}{2(\delta+\gamma)}\beta & -\dfrac{\gamma^2}{2(\delta+\gamma)}\\ \dfrac{\beta^2}{2(\delta+\gamma)} & -\dfrac{2\delta+\gamma}{2(\delta+\gamma)}\beta\end{bmatrix}$ and $b=\begin{bmatrix}\Big[-\dfrac{a}{2(\delta+\gamma)}+\dfrac{c}{2}\Big]\gamma+\dfrac{\gamma h}{2(\rho+\theta)}\big[e^{(\rho+\theta)(t-T)}-1\big]\\ \Big[\dfrac{a}{2(\delta+\gamma)}+\dfrac{c}{2}\Big]\beta+\dfrac{\beta h}{2(\rho+\theta)}\big[e^{(\rho+\theta)(t-T)}-1\big]\end{bmatrix}$.

The two eigenvalues of $A$ are and $m=(\rho/2)-\sqrt{\big((\rho/2)+\beta\big)\big((\rho/2)+\eta\beta\big)}$ and $\tilde{m}=(\rho/2)+\sqrt{\big((\rho/2)+\beta\big)\big((\rho/2)+\eta\beta\big)}$, where $\eta=\delta/(\delta+\gamma)$. The eigenvectors of $A$ can be obtained as

$$\Lambda=\begin{bmatrix}\dfrac{2\delta+\gamma}{\beta}+\dfrac{2(\delta+\gamma)}{\beta^2}m & \dfrac{2\delta+\gamma}{\beta}+\dfrac{2(\delta+\gamma)}{\beta^2}\tilde{m}\\ 1 & 1\end{bmatrix}.$$

Therefore, we have

$$\begin{aligned}\begin{bmatrix}\lambda_1\\ r\end{bmatrix}&=\Lambda\begin{bmatrix}e^{mt} & 0\\ 0 & e^{\tilde{m}t}\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix}-A^{-1}b\\ &=\begin{bmatrix}\frac{2\delta+\gamma}{\beta}e^{mt}+\frac{2(\delta+\gamma)}{\beta^2}me^{mt} & \frac{2\delta+\gamma}{\beta}e^{\tilde{m}t}+\frac{2(\delta+\gamma)}{\beta^2}\tilde{m}e^{\tilde{m}t}\\ e^{mt} & e^{\tilde{m}t}\end{bmatrix}\begin{bmatrix}k_1\\ k_2\end{bmatrix}\\ &\quad+\frac{1}{\Delta_1}\begin{bmatrix}(a-\delta c)\gamma+\frac{\delta\gamma h}{\rho+\theta}\big[1-e^{(\rho+\theta)(t-T)}\big]\\ (\rho+\beta)(a+c\delta)+\rho c\gamma+\frac{h}{\rho+\theta}(\beta\delta+\rho\delta+\rho\gamma)\big[e^{(\rho+\theta)(t-T)}-1\big]\end{bmatrix},\end{aligned}\tag{A.8}$$

where $\Delta_1=2\delta(\rho+\beta)+\rho\gamma$.

Considering the two boundary conditions $r(0)=r_0$ and $\lambda_1(T)=0$, we thus have

$$\begin{cases}r_0=k_1+k_2+\Delta_2,\\ 0=Q_1k_1+Q_2k_2.\end{cases}$$

Solving this linear equations, we get

$$k_1=\frac{Q_2}{Q_1-Q_2}(\Delta_2-r_0),\qquad k_2=\frac{Q_2}{Q_1-Q_2}(r_0-\Delta_2).$$

Substituting $k_1$ and $k_2$ into (A.4), gives

$$\lambda_1=\frac{1}{\Delta_1}\Big[(a-\delta c)\gamma+\frac{\delta\gamma h}{\rho+\theta}\big[e^{(\rho+\theta)(t-T)}-1\big]\Big],\tag{A.9}$$

$$r^*(t)=\frac{1}{\Delta_1}[(\rho+\beta)(a+\delta c)+\rho c\gamma]+c_1^*e^{mt}+c_2^*e^{\tilde{m}t}+\frac{h}{(\rho+\theta)\Delta_1}(\rho\delta+\beta\delta+\rho\gamma)\big[e^{(\rho+\theta)(t-T)}-1\big]\Big], \tag{A.10}$$

substituting (A.7), (A.9) and (A.10) into (A.2) gives the optimal price strategy (4.2).

Substituting (4.2) and (A.10) into (4.3), we get

$$\begin{aligned}\dot{I}+\theta I=&-a+\frac{\delta}{\Delta_1}[(\rho+\beta)(a+\delta c)+\rho c\gamma]+\frac{1-(2\delta+\gamma)}{2(\delta+\gamma)}(c_1^*e^{mt}+c_2^*e^{\tilde{m}t})\\&+\frac{h}{(\rho+\theta)\Delta_1}(\rho\delta+\beta\delta+\rho\gamma)\big[e^{(\rho+\theta)(t-T)}-1\big].\end{aligned} \tag{A.11}$$

Solving the first-order linear differential equation (A.11) by using the boundary condition $I(0)=I_0$, we obtain the optimal inventory policy (4.3). □

**Proof of Proposition 2.** Similar to optimization problem (4.1), the current-value Hamiltonian is given by (A.1). Applying general control theory, we have the following optimization conditions:

$$\frac{\partial H}{\partial p}=0\Leftrightarrow p=\frac{c+\lambda_2}{2}+\frac{a+\gamma r+\lambda_1\beta}{2(\delta+\gamma)}, \tag{A.12}$$

$$\dot{\lambda}_1=\rho\lambda_1-\frac{\partial H}{\partial r}=(\rho+\beta)\lambda_1-\gamma(p-c-\lambda_2), \tag{A.13}$$

$$\dot{\lambda}_2=\rho\lambda_2-\frac{\partial H}{\partial I}=(\rho+\theta)\lambda_2+h, \tag{A.14}$$

Substituting (A.12) into (A.13), (4.1) and (4.3), we get

$$\dot{\lambda}_1=\Big[\rho+\frac{2\beta+\gamma}{2(\delta+\gamma)}\beta\Big]\lambda_1+\frac{\gamma}{2}\lambda_2-\Big[\frac{a}{2(\delta+\gamma)}-\frac{c}{2}\Big]\gamma-\frac{\gamma^2}{2(\delta+\gamma)}r, \tag{A.15}$$

$$\dot{r}=\frac{\beta^2}{2(\delta+\gamma)}\lambda_1+\frac{\beta}{2}\lambda_2-\frac{2\delta+\gamma}{2(\delta+\gamma)}\beta r+\Big[\frac{a}{2(\delta+\gamma)}+\frac{c}{2}\Big]\beta, \tag{A.16}$$

$$\dot{I}=\frac{\beta}{2}\lambda_1+\frac{\delta+\gamma}{2}\lambda_2+\frac{\gamma}{2}r+\theta I-\frac{a}{2}. \tag{A.17}$$

Solving the first-order linear differential equations (A.14), we can obtain

$$\lambda_2=-\frac{h}{\rho+\theta}, \tag{A.18}$$

substituting (A.18) into (A.15) and (A.16), respectively, gives

$$\dot{\lambda}_1=\Big[\rho+\frac{2\delta+\gamma}{2(\delta+\gamma)}\beta\Big]\lambda_1-\frac{\gamma^2}{2(\delta+\gamma)}r-\frac{\gamma h}{2(\rho+\gamma)}-\Big[\frac{a}{2(\delta+\gamma)}-\frac{c}{2}\Big]\gamma, \tag{A.19}$$

$$\dot{r}=\frac{\beta^2}{2(\delta+\gamma)}\lambda_1-\frac{2\delta+\gamma}{2(\delta+\gamma)}\beta r-\frac{\beta h}{2(\rho+\theta)}+\Big[\frac{a}{2(\delta+\gamma)}+\frac{c}{2}\Big]\beta. \tag{A.20}$$

Thus, we derive $\begin{bmatrix} \dot{\lambda}_1 \\ \dot{r} \end{bmatrix} = A \begin{bmatrix} \lambda_1 \\ r \end{bmatrix} + \bar{b}$, where $A = \begin{bmatrix} \rho + \dfrac{2\beta+\gamma}{2(\delta+\gamma)}\beta & -\dfrac{\gamma^2}{2(\delta+\gamma)} \\ \dfrac{\beta^2}{2(\delta+\gamma)} & -\dfrac{2\delta+\gamma}{2(\delta+\gamma)}\beta \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} -\dfrac{\gamma h}{2(\rho+\gamma)} - \Big[\dfrac{a}{2(\delta+\gamma)} - \frac{c}{2}\Big]\gamma \\ -\dfrac{\beta h}{2(\rho+\theta)} + \Big[\dfrac{a}{2(\delta+\gamma)} + \dfrac{c}{2}\Big]\beta \end{bmatrix}$.

The two eigenvalues of $A$ are $m = (\rho/2) = \sqrt{\big((\rho/2)+\beta\big)\big((\rho/2)+\eta\beta\big)}$ and $\tilde{m} = (\rho/2) = \sqrt{((\rho/2)+\beta)((\rho/2)+\eta\beta)}$, where $\eta = (\delta/\delta+\gamma)$. The eigenvectors of $A$ can be obtained as

$$\Lambda = \begin{bmatrix} \dfrac{2\delta+\gamma}{\beta} + \dfrac{2(\delta+\gamma)}{\beta^2}m & \dfrac{2\delta+\gamma}{\beta} + \dfrac{2(\delta+\gamma)}{\beta^2}\tilde{m} \\ 1 & 1 \end{bmatrix}$$

Therefore, we have

$$\begin{aligned} \begin{bmatrix} \lambda_1 \\ r \end{bmatrix} &= \begin{bmatrix} e^{mt} & 0 \\ 0 & e^{\tilde{m}t} \end{bmatrix} \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} - A^{-1}\bar{b} \\ &= \begin{bmatrix} \dfrac{2\delta+\gamma}{\beta}e^{mt} + \dfrac{2(\delta+\gamma)}{\beta^2}me^{mt} & \dfrac{2\delta+\gamma}{\beta}e^{\tilde{m}t} + \dfrac{2(\delta+\gamma)}{\beta^2}\tilde{m}e^{\tilde{m}t} \\ e^{mt} & e^{\tilde{m}t} \end{bmatrix} \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} \\ &\quad + \frac{1}{\Delta_1} \begin{bmatrix} \frac{h}{\rho+\theta}\delta\gamma + (a-\delta c)\gamma \\ (\rho+\beta)(a+c\delta) + \rho c\gamma - (\beta\delta+\rho\delta+\beta\gamma)\frac{h}{\rho+\theta} \end{bmatrix}, \end{aligned}$$

where $\Delta_1 = 2\delta(\rho+\beta) + \rho\gamma = (\rho+\beta)\delta + (\delta+\gamma)(\rho+\eta\beta)$.

The two boundary conditions and gives $r(0) = r_0$ and $\lim_{t\to\infty} e^{-\rho t}\lambda_1(t) = 0$ gives $k_3 = r_0 - \frac{1}{\Delta_1}\Big[(\rho+\beta)(a+c\delta) + \rho c\gamma - \frac{h(\beta\delta+\rho\delta+\beta\gamma)}{\rho+\theta}\Big]$ and $k_4 = 0$. Hence, the optimal price and reference price paths can be given by

$$p_c^* = \overline{p}_c^{ss} - \frac{(2+\eta)\beta - \rho}{2(\rho+\theta)\Delta_1}\gamma h + (r_0 - \overline{p}_c^{ss})\Big(1 + \frac{m}{\beta}\Big)e^{mt}, \tag{A.21}$$

and

$$r_c^* = \overline{p}_c^{cc} + (r_0 - \overline{p}_c^{ss})e^{mt}, \tag{A.22}$$

where $\overline{p}_c^{ss} = \dfrac{1}{\Delta_1}\Big[(\rho+\beta)(a+c\delta) + \rho c\gamma - \dfrac{h(\beta\delta+\rho\delta+\beta\gamma)}{\rho+\theta}\Big]$.

Differentiating (A.21) with respective to $t$ yields $\partial p_c^*/\partial t = m(r_0 - \overline{p}_c^{ss})(1+m/\beta)e^{mt}$. And note that $1 + m/\beta > 0$ since $-\beta < m = (\rho/2) = \sqrt{\big((\rho/2)+\beta\big)\big((\rho/2)+\eta\beta\big)} < 0$ and $\eta = \delta/(\delta+\gamma) < 1$. Therefore, the positive or negative of $\partial p_c^*/\partial t$ depends on $r_0 - \overline{p}_c^{ss}$, i.e., when $r_0 > \overline{p}_c^{ss}$, $\partial p_c^*/\partial t < 0$, and when $r_0 < \overline{p}_c^{ss}$, $\partial p_c^*/\partial t > 0$. This indicates that $p_c^*$ converges to $\overline{p}_c^{ss}$ ($t \to \infty$). Moreover, if $r_0 = \overline{p}_c^{ss}$, then we have $p_c^* = \overline{p}_c^{ss} - [((2+\eta)\beta - \rho)/2(\rho+\theta)\Delta_1]\gamma h$ from (A.21). These implies that $[((2+\eta)\beta - \rho)/2(\rho+\theta)\Delta_1]\gamma h$, so the final optimal price path becomes (5.2).

Substituting (5.2) and (A.22) into (4.3) gives

$$\dot{I} = -\theta I + \Big[(\delta+\gamma)\Big(1+\frac{m}{\beta}\Big)-\gamma\Big](r_0-\overline{p}_c^{ss})e^{mt}-(a-\delta\overline{p}_c^{ss}). \tag{A.23}$$

Solving the first-order linear differential equations (A.23) by using boundary condition $I(0) = I_0$, we obtain the optimal inventory level (5.3). □

**Proof of Proposition 3.** Taking derivative of $\overline{p}_c^{ss}$ with respect to $\gamma$, we have

$$\frac{\partial \overline{p}_c^{ss}}{\partial \gamma} = -\frac{(\rho+\beta)\rho(a-\delta c)}{\Delta^2} - \frac{h\delta\rho(\rho+\beta)}{(\rho+\theta)\Delta^2} < 0,$$

we thus get the monotonicity of $\overline{p}_c^{ss}$ with $\gamma$. □

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