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A Finite-Source Inventory System with Service Facility, Multiple Vacations of Two Heterogeneous Servers

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Abstract

This paper deals a continuous review inventory system with a finite number of homogeneous sources of customers and multiple vacations of two heterogeneous servers. We have assumed that two heterogeneous servers who provide phase type services to customers. The inventory is replenished according to an (s, S) policy and the lead time follows an exponential distribution. The vacation times of both servers are assumed to be independent and identically distributed exponential random variables. The joint probability distribution of the inventory level, number of customers in the system and server status is obtained in the steady state. Some important performance measures are obtained and the optimality of an expected total cost rate is shown through numerical illustration.

Keywords: Finite source, phase-type distribution, heterogeneous servers, multiple vacations.

1. Introduction

Research on inventory systems with service facility has been considered by many authors over the last two decades. In this system, the demanded item is delivered to the customer after some service performed on it. Such situations occur when the items in the stock may require some random time for a service such as installation or preparation. As this causes the formation of a queue of demands, the inventory managers need to deal with the queue length, busy period of the server as well as the waiting time apart from the mean inventory level and reorder rate etc., to evaluate the system performance.

Berman et al. [8] introduced the concept of an inventory management system at a service facility which uses one item of inventory for each service provided. They assumed that both demand and service rates are deterministic and constant as such queues can form only during the stock-outs. They determined optimal order quantity that minimizes the total expected cost rate. Although the paper of Sigman and Simchi-Levi [25] published earlier than the paper of Berman et al. [8], the formers cited the work of later and hence we give the credit to the later. Sigman and Simchi-Levi [25] studied a single server inventory system in which the demands arrive according to a Poisson process, exponentially distributed replenishment time and arbitrary distribution for service times. Using light traffic heuristic method, they derived closed-form solution of the model. The interested reader may see Arivarignan et al. [3], Arivarignan and Sivakumar [4], Sivakumar and Arivarignan [28], Yadavalli et al. [30], Yadavalli et al. [31], Shophia Lawrence et al. [24]. Krishnamoorthy et al. [19] provides a partial survey of inventory systems with service facility.

Arivarignan et al. [5] considered a multi-server inventory system with service facility in which they assumed that the customers arrive according to a Markovian arrival process. They assumed that service times, lead times and the life time of an item were independent exponential distributions. Yadavalli et al. [30] extended this work by introducing the arrival of negative customers. They assumed that the negative customers arrive according to an independent Markovian arrival process. The service time, the lead time and the life time of an item were assumed to be independent exponential distributions. The customer who arrives during the stock-out period or when all the items are in service or when all the servers are busy entered into an orbit of infinite size and these customers compete for their service after a random amount of time. The time between two successive attempts has an exponential distribution.

In actual life the server is unavailable to the customers due to diverse causes. This includes, the server may be failed or may be employed in other works such as maintenance or serving secondary customers, or may just blend off. The aim of studying this model with vacation is, by utilizing the idle time of the server, by which the total average cost involved may be minimized. Applications arise naturally in call centers with multi-task employees, customized manufacturing, telecommunication and computer networks, maintenance activities, production and quality control problems, and so on.

Continuous review inventory systems with server vacation had been receiving little attention in the literature. The concept of server vacation in inventory with two servers was introduced by Danial and Ramanarayanan [9]. Danial and Ramanarayanan [10] studied an (s, S) inventory system in which the server takes a rest when the inventory level is zero. They assumed that the inter-arrival times between successive demands, lead times, and the rest times are assumed to follow arbitrary distributions. Krishnamoorthy and Narayanan [17] considered a production inventory system with server vacation. They assumed Markovian production process for production times and that service times for each customer had a phase-type distribution.

Sivakumar [27] considered an inventory system with retrial demands and multiple server vacation. He assumed independent exponential distributions for inter-demand times, lead times, inter-retrial times and server vacation times. He also assumed that all these events are mutually independent. He adopted a multiple vacation policy. Jayaraman et al. [15] considered a perishable inventory system with postponed demands in which the server takes multiple vacations. They assumed that demand time points form a Poisson process. The lifetime of each item, vacation time of the server and lead times follow independent exponential distributions. Padmavathi et al. [21] considered a continuous review stochastic (s, S) inventory system with Poisson demands and exponentially distributed lead time. They gave a comparative study of single and modified vacation policies.

In all the above mentioned articles related to inventory systems with multiple servers, the servers are assumed to be homogeneous. That is, the service rates are same for all the servers in the system. On the other hand, heterogeneity of service (the service rate at each server may be different) is a common feature of many real multi-server inventory systems. The heterogeneous service mechanisms are invaluable scheduling methods that allow customers to receive a different quality of service. Heterogeneous service is clearly a main feature of the operation of almost any manufacturing system. In queueing theory, the concept of heterogeneous service was studied by many authors, refer Morse [20], Saaty [23], Krishnakumar and Pavai Madheswari [16], Yue and Yue [32], Efrosinin and Sztrik [11], Krishnamoorthy and Sreenivasan [18], He and Xiuli-Chao [14], Ammar [2]. But in inventory theory, this concept was taken by Suganya et al. [29], in which they assumed MAP arrivals, phase type services, exponential lead times and exponential vacation times.

Most of the studies in the literature of inventory systems assumed the number sources that generate the primary customers to be infinite and then the flow of primary customers could be modelled by using Poisson process. However, when the customer population is of moderate size, it seems more appropriate that the inventory systems should be studied as a system with a finite source of customers. In these situations, it is often important to take into account the fact that the rate of generation of new primary customers decreases, as the number of customers in the system decreases. These types of arrival process can be modelled by using quasi-random input process.

The concept of finite population has been studied by many authors in queueing theory (see Falin and Templeton [13], Artalejo [6], Falin and Artalejo [12], Almási et al. [1], and Artalejo and Lopez-Herrero [7]). But in inventory systems, this concept was introduced by Sivakumar [26] in which he assumed that the arrival process follows a quasi-random input process, lifetime for an item, lead times and retrial times for the customers in the orbit follow independent exponential distributions. Shophia Lawrence et al. [24] studied the finite-source inventory system with service facility. They assumed that service times and lead times follow independent phase type distributions and the life time of an item follows an exponential distribution. Padmavathi et al. [22] studied the finite source inventory system with postponed demands and modified M vacation policy.

In this work, we focus on the case in which the population of demanding customers is finite, so that each individual customer generates his/her own flow of primary demands. The ordering doctrine is (s, S) policy with exponential lead time. The two heterogeneous servers can avail multiple vacations. The service times for two servers follow independent phase type distributions and vacation times of two servers follow independent exponential distributions with different parameters. The joint probability distribution of the number of customers in the system, inventory level and server status is obtained in the steady state case.

The rest of the paper is organized as follows: In section 2, we describe the mathematical model of the problem considered in this work. The analysis of the model is presented in section 3 and some key system performance measures are derived in section 4. We present some numerical studies in the final section. The following common notations are used throughout this paper: $\mathbf{e}$ denotes a column vector of 1's with appropriate dimension, $\mathbf{0}$ denotes a zero matrix of appropriate dimension, $\mathbf{I}$ denotes an identity matrix with the appropriate dimensions and E_i^j denotes $\{i, i+1, i+2, \ldots, j\}$.

2. Mathematical Model

We shall describe the concepts of phase-type distribution, so that the characterization of the inventory system can be defined.

Consider an absorbing Markov process with one absorbing state 0 and m transient states $1, 2, \ldots, m$. Let the initial probability distribution $(0, \beta)$ and the rate matrix

$$\tilde{T} = \begin{bmatrix} 0 & \mathbf{0} \\ T^0 & T \end{bmatrix}$$

The matrix T is a sub infinitesimal generator matrix, holding the transition probabilities among the m transient states, and T^0 contains the absorption probabilities into state 0 from the transient states. Clearly T^0 satisfies $T\mathbf{e} + \mathbf{T^0} = \mathbf{0}$. The mean of the phase-type distribution is given by $\mu = \beta(-T)^{-1}\mathbf{e}$. This phase-type distribution is represented by $(\beta, T)_m$.

Model We consider a continuous review (s, S) inventory system. Thus the maximum capacity of the inventory is S. Whenever the inventory level reaches a prefixed level, say $s(< S)$, an order for $Q(= S - s > s)$ items is placed, (This assumption $Q > s$ ensures that the replenished stock is always above s even if the stock is received after depletion of stock). We make the following assumptions:

- The demands are generated by a finite number of homogeneous sources and the demand time points form a quasi-random distribution with parameter α. That is, the probability that any particular source generates a demand in any interval $(t, t + dt)$ is $\alpha dt + o(dt)$ where $o(dt)/dt \to 0$ as $dt \to 0$ if the source is idle at time t, and zero if the source is in the service facility at time t, independently of the behaviour of any other sources.

- The time to deliver an order (or time for replenishing the stock) is assumed to have an exponential distribution with parameter $\theta(> 0)$.

- Customers are served under the first-come first-served (FCFS) discipline.

- We consider two heterogeneous servers. The service time of server-1 and server-2 are assumed to have independent phase-type distributions with representations $(\beta, U)_m$ and $(\delta, V)_n$ respectively. We write $U^0 = -Ue_m, V^0 = -Ve_n, \mu_1 = \beta(-U)^{-1}e_m$ and $\mu_2 = \delta(-V)^{-1}e_n$.

- Both servers can avail vacation whenever the inventory level reaches zero or the customer level reaches zero or both. At the end of a vacation period, the service commences if there is a positive inventory and at least one customer in the system; Otherwise, the server takes another vacation immediately and continues in the same manner until he finds both inventory level and the customer level are positive. This process holds good for both servers. The length of vacation time for i-th server is assumed to be independent and identically distributed as exponential with parameter γ_i, for $i = 1, 2$. These are independent of the length of service times, lead time and arrival process.

3. Analysis

Let $L(t), X(t), J_1(t), J_2(t)$ respectively, denote the on-hand inventory level, the number of customers in the system, and the phase of the first server and phase of the second server distribution at time t.

Further, let the status of the server $Y(t)$ be defined as follows:

$$Y(t) = \begin{cases} 0, \text{ if both the servers are on vacation} \\ 1, \text{ if the server 1 is busy and the server 2 is on vacation} \\ 2, \text{ if the server 1 is on vacation and the server 2 is busy} \\ 3, \text{ if both the servers are busy} \end{cases}$$

From our assumptions, it can be seen that the stochastic process $\{(L(t),\ X(t),\ Y(t),\ J_1(t),\ J_2(t)),\ t \geq 0\}$ is a Markov process with state space

$$\begin{aligned} \Omega = &\{(\ell, x, 0) : \ell \in E_0^S; x \in E_0^N\} \cup \\ &\{(\ell, x, 1, j_1) : \ell \in E_1^S; x \in E_1^N; j_1 \in E_1^m\} \cup \\ &\{(\ell, x, 2, j_2) : \ell \in E_1^S; x \in E_1^N; j_2 \in E_1^n\} \cup \\ &\{(\ell, x, 3, j_1, j_2) : \ell \in E_2^S; x \in E_2^N; j_1 \in E_1^m; j_2 \in E_1^n\} \end{aligned}$$

To introduce an order on the state space we define the following ordered sets

$$< \ell, x, y >= \begin{cases} ((\ell, x, y)); & \ell \in E_0^S,\, x \in E_0^N,\, y = 0 \\ ((\ell, x, y, 1), (\ell, x, y, 2), \ldots, (\ell, x, y, m)); & \ell \in E_1^S,\, x \in E_1^N,\, y = 1 \\ ((\ell, x, y, 1), (\ell, x, y, 2), \ldots, (\ell, x, y, n)); & \ell \in E_1^S,\, x \in E_1^N,\, y = 2 \\ ((\ell, x, y, j_1, 1), (\ell, x, y, j_1, 2), \ldots, (\ell, x, y, j_1, n)); & \ell \in E_2^S,\, x \in E_2^N,\, y = 3,\, j_1 \in E_1^m \end{cases}$$

$$<< \ell, x >> = \begin{cases} (< \ell, x, 0 >); & \ell \in E_0^S, x \in E_0^N, \\ (< \ell, x, 1 >, < \ell, x, 2 >); & \ell \in E_1^S, x \in E_1^N, \\ (< \ell, x, 3 >); & \ell \in E_2^S, x \in E_2^N, \end{cases}$$

$$\lll\ell\ggg = (<< \ell, 0 >>, << \ell, 1 >>, \ldots, << \ell, N >>)\,, \quad x = 0, 1, \ldots, S$$

Hence Ω is ordered as $(\lll 0\ggg, \lll 1\ggg, \ldots, \lll S\ggg)$. The infinitesimal generator of this process P can be written in a block partitioned form

$$P = ((P_{ij}))_{0,1,\ldots,S}$$

$$P = \begin{array}{c} \\ \lll 0\ggg \\ \lll 1\ggg \\ \lll 2\ggg \\ \lll 3\ggg \\ \vdots \\ \lll s\ggg \\ \lll s+1\ggg \\ \vdots \\ \lll s\ggg \end{array} \begin{array}{cccccccccccccc} \lll 0\ggg & \lll 1\ggg & \lll 2\ggg & \lll 3\ggg & \cdots & \lll s\ggg & \lll s+1\ggg & \cdots & \lll Q\ggg & \lll Q+1\ggg & \lll Q+2\ggg & \cdots & \lll S-1\ggg & \lll S\ggg \end{array}$$

$$\begin{pmatrix} A_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & C_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ B_1 & A_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & A_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & C_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_3 & A_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_3 & A_3 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \mathbf{0} & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_3 & A_3 \end{pmatrix}$$

The sub matrices are given in Appendix A.

3.1. Steady state analysis

It can be seen from the structure of P that the Markov Process $\{(L(t),\ X(t),\ Y(t),\ J_1(t),\ J_2(t)),\ t \geq 0\}$ on the finite state space Ω is irreducible. Hence the limiting distribution

$$\begin{aligned} \pi^{(i,k,y,j_1,j_2)} &= \lim_{t\to\infty} Pr\Big[L(t) = i, X(t) = k, Y(t) = j, J_1(t) = j_1, \\ &\qquad J_2(t) = j_2 | L(0), X(0), Y(0), J_1(0), J_2(0)\Big], \end{aligned}$$

exists.

Let π denote the steady state probability vector. It satisfies

$$\Pi P = \mathbf{0} \quad \text{and} \quad \Pi\mathbf{e} = 1. \tag{3.1}$$

The vector Π can be represented by

$$\Pi = (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(S)}).$$

where

$$\pi^{(i)} = (\pi^{(i,0)}, \pi^{(i,1)}, \ldots, \pi^{(i,N)}); \quad i = 0, 1, \ldots, S$$

$$\pi^{(i,k)} = \begin{cases} (\pi^{(i,k,0)}); & i = 0, 1, \ldots, S, \quad k = 0, 1, \ldots, N, \\ (\pi^{(i,k,1)}, \pi^{(i,k,2)}); & i = 1, 2, \ldots, S, \quad k = 1, 2, \ldots, N, \\ (\pi^{(i,k,3)}); & i = 2, 3, \ldots, S, \quad k = 2, 3, \ldots, N, \end{cases}$$

and

$$\pi^{(i,k,j)} = \begin{cases} (\pi^{(i,k,j)}); & i=0,1,\dots,S, \quad k=0,1,\dots,N, \quad j=0 \\ (\pi^{(i,k,j,1)}, \pi^{(i,k,j,2)}, \dots, \pi^{(i,k,j,m)}); & i=1,2,\dots,S, \quad k=1,2,\dots,N, \quad j=1 \\ (\pi^{(i,k,j,1)}, \pi^{(i,k,j,2)}, \dots, \pi^{(i,k,j,n)}); & i=1,2,\dots,S, \quad k=1,2,\dots,N, \quad j=2 \\ (\pi^{(i,k,j,j_1,1)}, \pi^{(i,k,j,j_1,2)}, \dots, \pi^{(i,k,j,j_1,n)}); & i=1,2,\dots,S, \quad k=1,2,\dots,N, \quad j=3, \\ & j_1=1,2,\dots,m \end{cases}$$

Lemma 1. The steady-state probability vector π corresponding to the infinitesimal generator matrix P is given by

$$\pi^{(i)} = \pi^{(Q)}\Omega_i, \quad i=0,1,\dots,S,$$

where

$$\Omega_i = \begin{cases} (-1)^{Q-i}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s-1}B_2A_1^{-1}B_1A_0^{-1}, & i=0, \\ (-1)^{Q-i}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s-1}B_2A_1^{-1}, & i=1, \\ (-1)^{Q-i}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s+1-i}, & i=2,3,\dots,s, \\ (-1)^{Q-i}(B_3A_3^{-1})^{Q-i}, & i=s+1,s+2,\dots,Q-1, \\ I, & i=Q, \\ (-1)^{Q-2}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s-1}B_2A_1^{-1}C_1A_3^{-1} & \\ -\sum_{j=i}^{S}(-1)^{2Q-j+1}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{S-j+1}C_2A_3^{-1}(B_3A_3^{-1})^{j-i}, & i=Q+1, \\ \sum_{j=i}^{S}(-1)^{2Q-j+1}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{S-j+1}C_2A_3^{-1}(B_3A_3^{-1})^{j-i}, & \\ & i=Q+2,Q+3\dots,S, \end{cases}$$

and $\pi^{(Q)}$ can be obtained by solving

$$\pi^{(Q)}\Bigg[\Big((-1)^{Q-2}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s-1}B_2A_1^{-1}C_1A_3^{-1} - \sum_{j=Q+2}^{S}\Big((-1)^{2Q-j+1}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{S-j+1}C_2A_3^{-1}(B_3A_3^{-1})^{j-(Q+1)}\Big)\Big)B_{(Q+1)} + A_Q + \Big((-1)^{Q}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s-1}B_2A_1^{-1}B_1A_0^{-1}\Big)C_0\Bigg] = \mathbf{0},$$

and

$$\pi^{(Q)}\Bigg[(-1)^{Q}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s-1}B_2A_1^{-1}B_1A_0^{-1}$$

$$+(-1)^{Q-1}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s-1}B_2A_1^{-1}$$
$$+\sum_{i=2}^{s}\Big((-1)^{Q-i}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s+1-i}\Big)$$
$$+\sum_{i=s+1}^{Q-1}((-1)^{Q-i}(B_3A_3^{-1})^{Q-i})+I+(-1)^{Q-2}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{s-1}B_2A_1^{-1}C_1A_3^{-1}$$
$$-\sum_{j=Q+2}^{S}\Big((-1)^{2Q-j+1}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{S-j+1}C_2A_3^{-1}(B_3A_3^{-1})^{j-(Q+1)}\Big)$$
$$+\sum_{i=Q+2}^{S}\sum_{j=i}^{S}\Big((-1)^{2Q-j+1}(B_3A_3^{-1})^{Q-(s+1)}(B_3A_2^{-1})^{S-j+1}C_2A_3^{-1}(B_3A_3^{-1})^{j-i}\Big)\Bigg]\mathbf{e}=1.$$

Proof. The first equation of equation (3.1) yields the following set of equations :

$$\pi^{(i+1)}B_1+\pi^{(i)}A_0=\mathbf{0}, \quad i=0, \tag{3.2}$$
$$\pi^{(i+1)}B_2+\pi^{(i)}A_1=\mathbf{0}, \quad i=1, \tag{3.3}$$
$$\pi^{(i+1)}B_3+\pi^{(i)}A_2=\mathbf{0}, \quad i=2,3,\ldots,s, \tag{3.4}$$
$$\pi^{(i+1)}B_3+\pi^{(i)}A_3=\mathbf{0}, \quad i=s+1,s+2,\ldots,Q-1, \tag{3.5}$$
$$\pi^{(i+1)}B_3+\pi^{(i)}A_3+\pi^{(i-Q)}C_0=\mathbf{0}, \quad i=Q, \tag{3.6}$$
$$\pi^{(i+1)}B_3+\pi^{(i)}A_3+\pi^{(i-Q)}C_1=\mathbf{0}, \quad i=Q+1, \tag{3.7}$$
$$\pi^{(i+1)}B_3+\pi^{(i)}A_3+\pi^{(i-Q)}C_2=\mathbf{0}, \quad i=Q+2,Q+3\ldots,S-1, \tag{3.8}$$
$$\pi^{(i)}A_3+\pi^{(i-Q)}C_2=\mathbf{0}, \quad i=S. \tag{3.9}$$

Solving the above system of equations (except 3.6) recursively and using the normalizing condition, we get the stated result. □

4. System Performance Measures

In this section, we calculate some system performance measures useful in qualitative interpretation of the model under study. We shall use the term $\pi^{(i,k,\ell)}$ to represent the steady state probability vector when the inventory level is i, the number of customers in the system is k and the server status is j and with all other phases.

4.1. Expected inventory level

Let ζ_I denote the expected inventory level in the steady state. The expected inventory level is given by

$$\zeta_I = \sum_{i=1}^{S} i\pi^{(i)}\mathbf{e}. \tag{4.1}$$

4.2. Expected number of customers in the system

Let ζ_P denote the expected number of customers in the system. Since $\pi^{(i,k)}$ is a vector of probabilities with the inventory level is i and k customers in the system, the expected number of customers in the system ζ_P in the steady state is given by

$$\zeta_P = \sum_{i=0}^{S} \sum_{k=1}^{N} k\pi^{(i,k)}\mathbf{e}. \tag{4.2}$$

4.3. Expected reorder level

Let ζ_R denote the expected reorder level in the steady-state. A reorder is placed when the inventory level drops from $s+1$ to s. It may occur when the inventory level is $s+1$ and the server completes a service for a customer. Hence, we get

$$\begin{aligned} \zeta_R &= U^0\pi^{(s+1,1,1)}\mathbf{e} + \sum_{k=2}^{N} U^0\beta\pi^{(s+1,k,1)}\mathbf{e} \\ &+V^0\pi^{(s+1,1,2)}\mathbf{e} + \sum_{k=2}^{N} V^0\delta\pi^{(s+1,k,2)} \\ &+\sum_{i=2}^{N}(U^0\beta + V^0\delta)\pi^{(s+1,k,3)}\mathbf{e}. \end{aligned} \tag{4.3}$$

4.4. Expected waiting time of customer

Let $E(W)$ denote the mean waiting time of the customer. Then by little's formula

$$E(W) = \frac{\zeta_P}{\alpha} \tag{4.4}$$

4.5. Total expected cost

Let $TC(s,S)$ denote the long-run expected cost rate under the following cost structure:

c_s : Set up cost per order.

c_h : The inventory carrying cost per unit item per unit time.

c_o : Waiting cost of a customer in the system per unit time.

Then

$$TC(s,S) = c_h\zeta_I + c_s\zeta_R + c_o\zeta_P.$$

Since the computation of the π's involve recursive equations, it is difficult to study the qualitative behaviour of the total expected cost rate analytically. However, we present the following numerical analysis to demonstrate the computability of the results derived in our work.

5. Numerical Analysis

In this section, we have used 'simple' numerical search procedures to find the "local" optimal values by considering a small set of integer values for the decision variables.

For service time distribution of each of the servers, we consider the following three PH distributions.

1. **Exponential** (EXP)

$$D_0 = (-1) \quad D_1 = (1)$$

2. **Erlang** (ERL)

$$D_0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

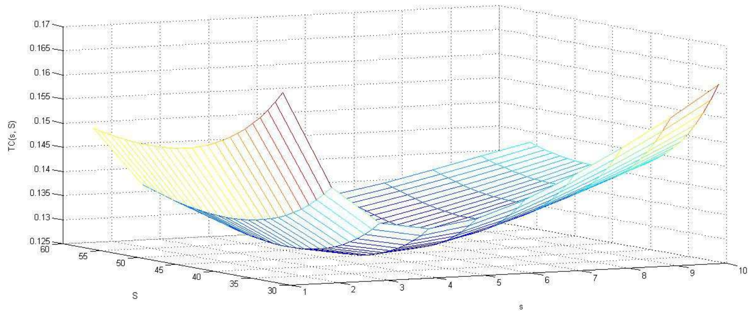
3. **Hyper-exponential** (HEX)

$$D_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \qquad D_1 = \begin{pmatrix} 9 & 1 \\ 0.9 & 0.1 \end{pmatrix}$$

These three phase type distributions will be normalized so as to have a specific service rates μ_1 and μ_2. These processes are special cases of renewal processes and the correlation between service times is zero.

A three dimensional plot of $TC(s, S)$ for Erlang service for both servers is presented in Figure 1 which shows the convex nature of the cost function. We note that all the tables in this chapter, upper entries in each cell give the optimal values S^* and s^* and the lower entry gives the corresponding optimal cost rate. We also note that, in all the table we interpret $EXP - i$ as exponential service time distribution for server $i, i = 1, 2$. That is, first three letter denotes the distributions (EXP, ERL and HEX) and the $-i$ denotes the server.

Example 1. In Table 1, we provide the optimum values, S^* and s^*, that minimizes the expected total cost rate for each of the three PHs of two heterogeneous service times ($PH1$ for server 1, $PH2$ for server 2). The effect of variations in the holding cost c_h, set up cost c_s and waiting cost of a customer c_o on the optimal values are tabulated. We fix the system parameter values as $\theta = 0.9$, $\gamma_1 = 0.25$, $\gamma_2 = 1.5$, $\alpha = 0.2$, $\mu_1 = 0.667$, $\mu_2 = 0.444$, $N = 20$. The key observations are listed below:



$$\alpha = 0.2; \theta = 0.9; \mu_1 = 1.5; \mu_2 = 2.25; \gamma_1 = 0.25; \gamma_2 = 1.5; N = 20$$
$$c_h = 0.003; c_s = 1.7; c_o = 0.4;;$$

Figure 1: A three dimensional plot for convexity of total expected cost rate

- As is to be expected S is a non decreasing function of c_s and c_o. This is because, if the replenishment cost increases, the manager has to maintain high inventory so as to avoid frequent ordering. If the waiting time increases, one has to maintain a large inventory to reduce the number of waiting customers in the system.

- Similarly S is a non increasing function of c_h. This is to be expected, since the holding cost increases, the managers have to maintain low stock.

- We also note that s is a non increasing function of c_h, c_s and non decreasing function of c_o

Example 2. In this example we analyse the sensitivity of the parameters for fixing the cost values by the tables 2 - 7. For all the models, we fix the costs as follows. $c_h = 0.003; c_s = 1.7; c_o = 0.4$. The key observations are listed below:

- If α increases, then the optimal inventory level and the optimal reorder level also increase monotonically. Due to this increasing of inventory level and reorder level, holding cost affects the total cost. So that the total cost value increases when α increases. Similar behaviour is observed for θ also.

- The total cost increases as γ_1 and γ_2 increase.

- If the service rate for server 1 (μ_1) increases, then s, S and TC decrease. But the service rate for server 2 increases, only the total cost decreases. It does not affect the inventory level because we give the higher service rate for server 1. The reason will be provided in the next example.

Table 1: Effect of costs on the optimal values

c_h	c_s	c_o	PH Services								
			$EXP-1$			$ERL-1$			$HEX-1$		
			$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$
0.0025	1.65	0.35	45 5	45 5	45 5	45 5	45 5	45 5	45 5	46 6	45 5
			0.117585	0.113504	0.116647	0.113162	0.112427	0.114762	0.119007	0.116023	0.122704
		0.4	45 5	45 5	46 6	45 5	45 5	46 6	46 6	46 6	46 6
			0.11882	0.114156	0.1175	0.113763	0.112923	0.115327	0.120126	0.11668	0.124601
		0.45	45 6	45 6	46 6	45 6	45 5	46 6	46 6	46 6	46 6
			0.119794	0.114545	0.118258	0.114227	0.113418	0.115812	0.121215	0.117337	0.126254
	1.7	0.35	45 5	45 5	46 5	45 5	45 5	46 5	46 5	46 6	46 5
			0.118964	0.114885	0.118001	0.114543	0.11381	0.11612	0.120353	0.117403	0.124055
		0.4	46 5	46 5	46 6	46 5	45 5	46 6	46 6	46 6	46 6
			0.120185	0.115525	0.118881	0.11514	0.114306	0.11671	0.121505	0.11806	0.125978
		0.45	46 6	46 6	46 6	46 6	45 5	46 6	47 6	47 6	47 6
			0.121179	0.115935	0.11964	0.115624	0.114801	0.117195	0.12259	0.118709	0.127628
	1.75	0.35	46 5	46 5	46 5	46 5	46 5	46 5	46 5	47 6	46 5
			0.120311	0.116236	0.119344	0.115901	0.115185	0.117466	0.121694	0.118769	0.125394
		0.4	46 5	46 5	47 6	46 5	46 5	47 6	47 6	47 6	47 6
			0.121531	0.116872	0.120244	0.116488	0.115668	0.118079	0.12286	0.119412	0.127333
		0.45	47 6	47 6	47 6	46 6	46 5	47 6	47 6	47 6	47 6
			0.122557	0.117318	0.120989	0.117009	0.116151	0.118553	0.123936	0.120055	0.128972
0.003	1.65	0.35	41 5	41 5	42 5	41 5	41 5	42 5	42 5	42 5	42 5
			0.129796	0.125699	0.128961	0.125324	0.124514	0.127056	0.131351	0.128507	0.13503
		0.4	42 5	42 5	42 5	42 5	41 5	42 6	42 6	42 6	42 5
			0.131093	0.126413	0.130133	0.125991	0.125064	0.127938	0.1328	0.129366	0.137077
		0.45	42 6	42 6	42 6	42 5	41 5	42 6	43 6	43 6	42 6
			0.132362	0.127096	0.130953	0.126641	0.125615	0.128474	0.13395	0.13008	0.138985
	1.7	0.35	42 5	42 5	42 5	42 5	42 5	42 5	42 5	42 5	42 5
			0.131293	0.127201	0.130448	0.126834	0.126044	0.128546	0.132835	0.129992	0.136512
		0.4	42 5	42 5	43 5	42 5	42 5	43 5	43 6	43 6	43 5
			0.132583	0.127905	0.131613	0.127484	0.126579	0.129441	0.134306	0.130869	0.138555
		0.45	42 5	42 5	43 6	42 5	42 5	43 5	43 6	43 6	43 6
			0.133873	0.128609	0.132453	0.128134	0.127115	0.129982	0.13544	0.131571	0.140475
	1.75	0.35	42 5	42 5	43 5	42 5	42 5	43 5	43 5	43 5	43 5
			0.132783	0.128692	0.131914	0.128326	0.127538	0.130017	0.134293	0.131443	0.137976
		0.4	43 5	43 5	43 5	43 5	42 5	43 5	43 5	43 6	43 5
			0.134062	0.129388	0.133062	0.128976	0.128074	0.130892	0.135784	0.13236	0.139998
		0.45	43 5	43 5	43 6	43 5	42 5	43 6	43 6	44 6	43 6
			0.135333	0.130073	0.133946	0.129609	0.128609	0.131477	0.13693	0.133058	0.141963
0.0035	1.65	0.35	39 5	39 5	39 5	39 5	38 5	39 5	39 5	39 5	39 5
			0.141225	0.137117	0.14049	0.136721	0.135851	0.138566	0.14291	0.14009	0.146568
		0.4	39 5	39 5	39 5	39 5	39 5	39 5	39 5	40 6	39 5
			0.142578	0.137882	0.141743	0.137428	0.136446	0.139541	0.144512	0.141277	0.148697
		0.45	39 5	39 5	40 6	39 5	39 5	40 6	40 6	40 6	40 5
			0.143932	0.138647	0.142867	0.138135	0.137029	0.140365	0.145891	0.142032	0.150822
	1.7	0.35	39 5	39 5	39 5	39 5	39 5	40 5	40 5	40 5	39 5
			0.142845	0.138739	0.142108	0.138344	0.13749	0.140187	0.144521	0.141692	0.148179
		0.4	39 5	39 5	40 5	39 5	39 5	40 6	40 5	40 5	40 5
			0.144199	0.139504	0.143333	0.139051	0.138072	0.141138	0.146093	0.142858	0.150286
		0.45	40 5	40 5	40 6	39 5	39 5	40 6	40 6	40 6	40 5
			0.145543	0.140264	0.144491	0.139758	0.138655	0.141992	0.147511	0.143654	0.152387
	1.75	0.35	40 5	40 5	40 5	39 5	39 5	40 6	40 5	40 5	40 5
			0.144455	0.140353	0.143681	0.139967	0.139115	0.141765	0.146088	0.143261	0.149751
		0.4	40 5	40 5	40 5	40 5	39 5	40 5	40 5	40 5	40 5
			0.145786	0.141097	0.144905	0.140654	0.139698	0.142713	0.14766	0.144427	0.151851
		0.45	40 5	40 5	41 6	40 5	40 5	41 6	41 6	41 6	40 5
			0.147117	0.14184	0.146103	0.141341	0.140266	0.143613	0.149113	0.145251	0.153952

Table 2: Effect of arrival rate on optimal values

$\gamma_1 = 0.25; \gamma_2 = 1.5; \theta = 0.9; \mu_1 = 0.667; \mu_2 = 0.444$

α	PH Services								
	$EXP-1$			$ERL-1$			$HEX-1$		
	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$
0.15	40 4	42 5	42 5	41 4	42 5	40 4	42 5	42 5	42 5
	0.181595	0.128161	0.15291	0.16783	0.138913	0.179879	0.127318	0.126065	0.132395
0.2	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132583	0.127905	0.131613	0.134306	0.130869	0.138555	0.127484	0.126579	0.129441
0.25	42 5	42 5	43 5	43 6	43 6	43 6	42 5	42 5	43 6
	0.12397	0.128371	0.130361	0.131268	0.130778	0.132768	0.127921	0.126916	0.19725

Table 3: Influence of vacation rate of the second server on the optimal values
$\gamma_1 = 0.25; \theta = 0.9; \alpha = 0.2; \mu_1 = 0.667; \mu_2 = 0.444$

γ_2	PH Services								
	$EXP-1$			$ERL-1$			$HEX-1$		
	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$
1.4	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132582	0.127904	0.131612	0.134305	0.130868	0.138545	0.127484	0.126579	0.129440
1.5	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132583	0.137905	0.131613	0.134306	0.130869	0.138555	0.127484	0.126579	0.129441
1.6	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132584	0.127905	0.131615	0.134307	0.130870	0.138565	0.127484	0.126579	0.129441

Table 4: Sensitivity of γ_1 on the optimal values
$\gamma_2 = 1.5; \theta = 0.9\alpha = 0.2; \mu_1 = 0.667; \mu_2 = 0.444$

γ_1	PH Services								
	$EXP-1$			$ERL-1$			$HEX-1$		
	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$
0.24	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132575	0.1279	0.131607	0.134297	0.130863	0.138528	0.12748	0.126576	0.129436
0.25	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132583	0.127905	0.131613	0.134306	0.130869	0.138555	0.127484	0.126579	0.129441
0.26	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132591	0.127909	0.13162	0.134315	0.130875	0.138581	0.127487	0.126582	0.129445

Table 5: Influence of the replenishment rate on the optimal values
$\gamma_1 = 0.25; \gamma_2 = 1.5; \alpha = 0.2; \mu_1 = 0.667; \mu_2 = 0.444$

θ	PH Services								
	$EXP-1$			$ERL-1$			$HEX-1$		
	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$
0.85	43 6	43 6	43 6	43 6	43 6	43 6	43 6	42 5	43 6
	0.13355	0.128866	0.132437	0.135167	0.1318	0.139633	0.128511	0.127573	0.130231
0.9	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132583	0.127905	0.131613	0.134306	0.130869	0.138555	0.127484	0.126579	0.129441
0.95	42 5	42 5	42 5	42 5	42 5	42 5	42 5	41 5	42 5
	0.131728	0.127039	0.130617	0.133277	0.129931	0.137526	0.126679	0.125848	0.128428

Example 3. In this example, we explain the effect of service rates on the system performance measures through figure (2). We interpret the figure as follows. The values on the x axis represent service rate of server 1, if $\mu_1 > \mu_2$, and values of the service rate

Table 6: Effect of second server service rate on the optimal values
$\gamma_1 = 0.25; \gamma_2 = 1.5; \theta = 0.9; \alpha = 0.2; \mu_1 = 0.667;$

μ_2	PH Services								
	$EXP-1$			$ERL-1$			$HEX-1$		
	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$
0.442	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132391	0.127785	0.131448	0.134146	0.130750	0.138332	0.127362	0.126462	0.129308
0.444	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132583	0.127905	0.131613	0.134306	0.130869	0.138555	0.127484	0.126579	0.129441
0.446	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132778	0.128026	0.131782	0.134468	0.130989	0.138782	0.127607	0.126698	0.129575

Table 7: Sensitivity of μ_1 on the optimal values
$\gamma_1 = 0.25; \gamma_2 = 1.5; \theta = 0.9; \alpha = 0.2; \mu_2 = 0.444;$

μ_1	PH Services								
	$EXP-1$			$ERL-1$			$HEX-1$		
	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$	$EXP-2$	$ERL-2$	$HEX-2$
0.645	42 5	42 5	42 5	42 5	42 6	42 5	42 5	41 5	42 5
	0.130627	0.126643	0.130111	0.132339	0.129499	0.136279	0.126273	0.125352	0.128197
0.667	42 5	42 5	43 5	43 6	43 6	43 5	42 5	42 5	43 5
	0.132583	0.127905	0.131613	0.134306	0.130869	0.138555	0.127484	0.126579	0.129441
0.689	43 5	43 5	43 6	43 6	43 6	43 5	42 5	42 5	43 6
	0.134808	0.129258	0.133215	0.136501	0.132371	0.141083	0.128786	0.127887	0.130688

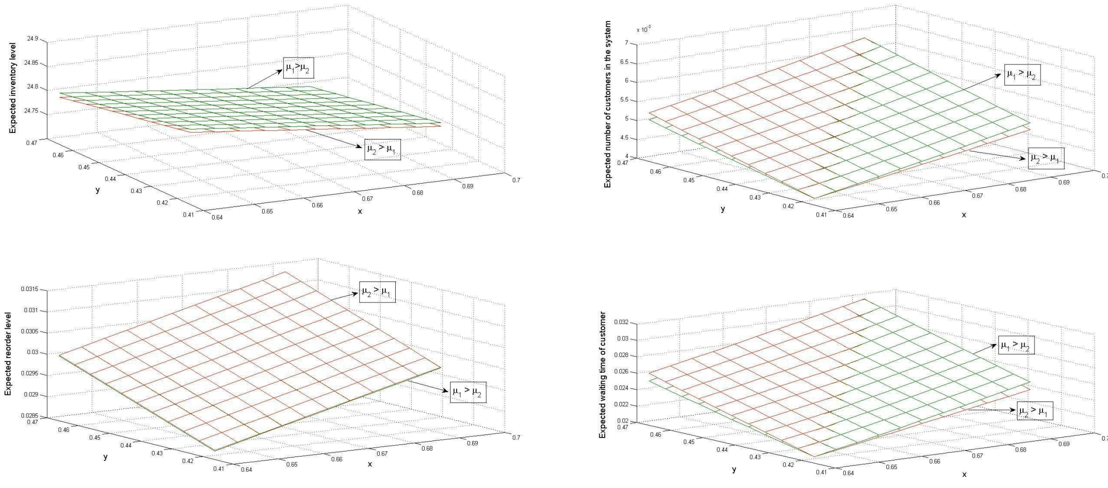
of server 2, if $\mu_1 < \mu_2$. Similarly, the values on the y axis represent the service rate of server 2 if $\mu_1 < \mu_2$, and values of the service rate of server 2, if $\mu_1 > \mu_2$. From figure (2) we observe the following:

- If μ_1 and μ_2 increases then the expected inventory level decreases for both cases ($\mu_1 < \mu_2$ and $\mu_1 > \mu_2$).
- If μ_1 and μ_2 increases then the expected number of customers in the system, the expected reorder level and the expected waiting time of customer increase for both cases ($\mu_1 < \mu_2$ and $\mu_1 > \mu_2$).

Example 4. We evaluate the impact of the service time of both servers on expected waiting time. The corresponding graphs are depicted in figure (3). Here, we assume that the service time distribution for server 1 is fixed and vary the service time distribution for server 2. Other parameters are fixed as $\alpha = 0.2; \theta = 0.9; \gamma_1 = 0.25; \gamma_2 = 1.5; N = 20;$

When the service time for server 1 follows exponential distribution, we observe the following:

- $E(W)$ is low when the service time for server 2 follows Erlang distribution and it is high when the service time for server 2 follows exponential distribution.
- When the service time for server 2 follows either of the three distributions, i.e., exponential, Erlang and hyper-exponential, the $E(W)$ is decreasing as the reorder level s is increasing.



$$\alpha = 0.2; \theta = 0.9; \gamma_1 = 0.25; \gamma_2 = 1.5; N = 20$$

$$c_h = 0.003; c_s = 1.7; c_o = 0.4;;$$

Figure 2: Effect of service rates for both servers on the system performance measures

- When the service time for server 2 follows either of the three distributions, i.e., exponential, Erlang and hyper-exponential, the $E(W)$ is decreasing as the maximum inventory level is increasing.

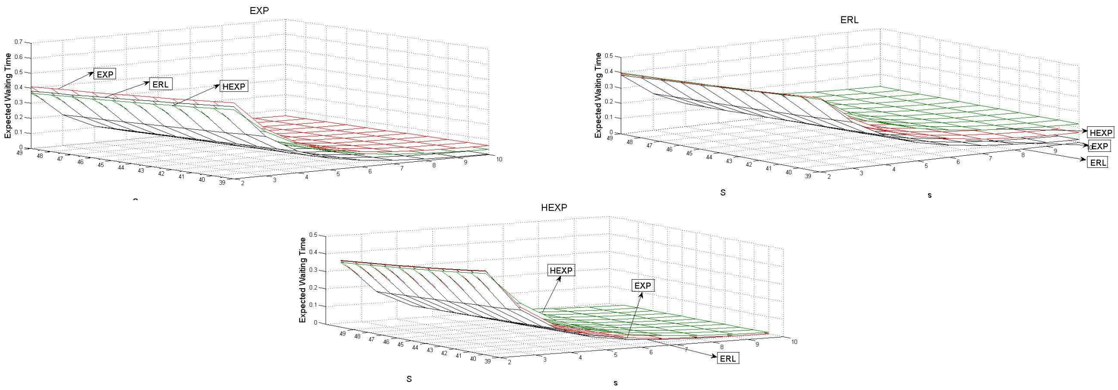
When the service time for server 1 follows Erlang distribution, we observe the following:

- $E(W)$ is low when the service time for server 2 follows Erlang distribution and it is high when the service time for server 2 follows hyper-exponential distribution.
- When the service time for server 2 follows exponential distribution, the $E(W)$ is decreasing as the reorder level s is increasing.
- When the service time for server 2 follows Erlang distribution, the $E(W)$ is increasing at a higher rate as the reorder level s is increasing.

And when the service time for server 1 follows hyper-exponential distribution, the $E(W)$ behaves as in the case of Erlang distribution.

6. Conclusion

In this paper, we have analyzed a finite source inventory system with two heterogeneous servers who avail multiple vacations. The demands are generated by a finite number of homogeneous sources and the service times have phase type distributions for each server. Lead times and vacation durations of each server are distributed exponentially. The major contribution made in this paper is to allow vacation for each server



$$\alpha = 0.2; \theta = 0.9; \mu_1 = 0.667; \mu_2 = 0.444; \gamma_1 = 0.25; \gamma_2 = 1.5; N = 20$$
$$c_h = 0.003; c_s = 1.7; c_o = 0.4;;$$

Figure 3: Effect of Expected waiting time on S and s for both servers

and also to allow multiple vacations. Unlike in the queueing theory context, the vacation starts not only when the customer level becomes zero, but also when the inventory is depleted. At the end of vacation, even if there is at least one customer in the system, another vacation will start if there are no items in the stock. Since this model includes these real time aspects it has a wider scope for application. The stability of this system is analyzed by looking at the continuous-time Markov chain associated with this process. The stationary distribution of the system state is obtained. A few measures of performance are computed. Using numerical illustrations on some specified collection of parameters, we have studied the sensitivity of various cost on the optimal values, the sensitivity of the parameter for fixing the cost values, the effect of service rates on the system performance measures and the impact of service time of both servers on expected waiting time.

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Appendix A

$$[C_0]_{k\ell} = \begin{cases} \theta, & \ell = k, \quad k = 0 \\ E_0^{(1)}, & \ell = k, \quad k = 1 \\ E_0^{(2)}, & \ell = k, \quad k = 2, 3, \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$E_0^{(1)} = \begin{pmatrix} \theta & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$E_0^{(2)} = \begin{pmatrix} \theta & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$[C_1]_{k\ell} = \begin{cases} \theta, & \ell = k, \quad k = 0 \\ E_0^{(3)}, & \ell = k, \quad k = 1 \\ E_0^{(4)}, & \ell = k, \quad k = 2, 3, \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$E_0^{(3)} = \begin{pmatrix} \theta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \theta I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \theta I_n \end{pmatrix}$$

$$E_0^{(4)} = \begin{pmatrix} \theta & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \theta I_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \theta I_n & \mathbf{0} \end{pmatrix}$$

$$[C_2]_{k\ell} = \begin{cases} \theta, & \ell = k, \quad k = 0 \\ E_0^{(3)}, & \ell = k, \quad k = 1 \\ E_0^{(5)}, & \ell = k, \quad k = 2, 3, \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$E_0^{(5)} = \begin{pmatrix} \theta & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \theta I_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \theta I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \theta (I_m \otimes I_n) \end{pmatrix}$$

$$[B_1]_{k\ell} = \begin{cases} E_1^{(1)}, & \ell = k - 1, \quad k = 1, 2, \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$E_1^{(1)} = \begin{pmatrix} \mathbf{0} \\ U^0 \\ V^0 \end{pmatrix}$$

$$[B_2]_{k\ell} = \begin{cases} E_1^{(1)}, & \ell = k-1, \quad k = 1 \\ E_1^{(2)}, & \ell = k-1, \quad k = 2, 3, \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$E_1^{(2)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & U^0\beta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & V^0\delta \\ \mathbf{0} & I_m \otimes V^0 & U^0 \otimes I_n \end{pmatrix}$$

$$[B_3]_{k\ell} = \begin{cases} E_1^{(1)}, & \ell = k-1, \quad k = 1 \\ E_1^{(2)}, & \ell = k-1, \quad k = 2 \\ E_1^{(3)}, & \ell = k-1, \quad k = 3, 4, \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$E_1^{(3)} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & U^0\beta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & V^0\delta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (U^0\beta \oplus V^0\delta) \end{pmatrix}$$

$$[A_0]_{k\ell} = \begin{cases} (N-k)\alpha, & \ell = k+1, \quad k = 0, 1, \ldots, N-1 \\ -((N-k)\alpha + \theta), & \ell = k, \quad k = 0, 1, \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$[A_1]_{k\ell} = \begin{cases} F_1^{(1)}, & \ell = k+1, \quad k = 0 \\ F_1^{(k+1)}, & \ell = k+1, \quad k = 1, 2 \ldots, N-1 \\ -(N\alpha + \theta), & \ell = k, \quad k = 0 \\ E_2^{(k)}, & \ell = k, \quad k = 1, 2 \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$F_1^{(1)} = \begin{pmatrix} N\alpha & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$F_1^{(k+1)} = \begin{pmatrix} (N-k)\alpha & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (N-k)\alpha I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (N-k)\alpha I_n \end{pmatrix}$$

$$E_2^{(k)} = \begin{pmatrix} -((N-k)\alpha + \theta + \gamma_1 + \gamma_2) & \gamma_1\beta & \gamma_2\delta \\ \mathbf{0} & U - ((N-k)\alpha + \theta) I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & V - ((N-k)\alpha + \theta) I_n \end{pmatrix}$$

$$[A_2]_{k\ell} = \begin{cases} F_1^{(2)}, & \ell = k+1, \quad k = 0 \\ F_2^{(1)}, & \ell = k+1, \quad k = 1 \\ F_2^{(k)}, & \ell = k+1, \quad k = 2, 3 \ldots, N-1 \\ -(N\alpha + \theta), & \ell = k, \quad k = 0 \\ E_2^{(1)}, & \ell = 1, \quad k = 1 \\ E_3^{(k-1)}, & \ell = k, \quad k = 2, 3 \ldots, N \\ \mathbf{0}, & \text{Otherwise} \end{cases}$$

$$F_2^{(1)} = \begin{pmatrix} (N-k)\alpha & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (N-k)\alpha I_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (N-k)\alpha I_n & \mathbf{0} \end{pmatrix}$$

$$F_2^{(k)} = \begin{pmatrix} (N-k)\alpha & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (N-k)\alpha I_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (N-k)\alpha I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (N-k)\alpha(I_m \otimes I_n) \end{pmatrix}$$

$$E_3^{(k-1)} = \begin{pmatrix} -((N-k)\alpha+\theta+\gamma_1+\gamma_2) & \gamma_1\beta & \gamma_2\delta & \mathbf{0} \\ \mathbf{0} & U-((N-k)\alpha+\theta+\gamma_2)I_m & \mathbf{0} & \gamma_2\delta \\ \mathbf{0} & \mathbf{0} & V-((N-k)\alpha+\theta+\gamma_1)I_n & \gamma_1\beta \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (U\oplus V)-((N-k)\alpha+\theta)(I_m\otimes I_n) \end{pmatrix}$$

$$A_3 = A_2 + \theta I$$

Table 8: The sub matrices and their dimensions

Matrices	Dimensions
A_0	$(N+1) \times (N+1)$
A_1	$(N+1+Nm+Nn) \times (N+1+Nm+Nn)$
A_2, A_3, B_3, C_2	$(N+1+Nm+Nn+(N-1)mn) \times (N+1+Nm+Nn+(N-1)mn)$
B_1	$(N+1+Nm+Nn) \times (N+1)$
B_2	$(N+1+Nm+Nn+(N-1)mn) \times (N+1+Nm+Nn)$
C_0	$(N+1) \times (N+1+Nm+Nn+(N-1)mn)$
C_1	$(N+1+Nm+Nn) \times (N+1+Nm+Nn+(N-1)mn)$

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